# The AP Calculus AB <br> Survival Manual 



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## Introduction

Do you need to get through a calculus course, preferably with a decent grade? After surviving basic calculus classes in three different countries I decided to write this manual to help other students. Hopefully this e-book will help you understand what is going on in your class, or help you review if you took calculus a while ago. You wouldn't think so when you are in the middle of your course, but if you take calculus out of the classroom it is really quite awesome. I am forever grateful to Kamex for showing me this, and for teaching me about infinitely small things.

This is a continuation of the free e-book What is Calculus? - A Bedtime Story, available at the website that this book was downloaded from. If at all possible you will want to read that first, but if your course has already started you may not have time. A brief summary is provided in the first chapter. Please read it.

## Survival Tips

1. Don't miss class. But if it is unavoidable, get notes, handouts, and important class news from a friend afterwards.
2. If you don't have a friend, make one. Common conversation starters in a calculus class are "I don't get any of this stuff," and "I hope the test isn't too hard."
3. Study with a friend. It makes calculus less boring, and you'll either learn from your friend or learn by explaining things to your friend, or both ways.
4. Read over your class notes when you get home. Math often takes a while to sink in, and you can't always expect to understand everything right away as your teacher explains it. If you don't have notes or can't follow them read this e-book.
5. When you read your notes or a book, cover up the solution to the examples and see if you can find the answer yourself. Even if you can't, you'll be better prepared to understand the solution when you read it.
6. Ask a question during class. Several other students are likely to have the same question, but are afraid to ask.
7. If you have additional questions, talk to your teacher after class. Ask specific, focused questions rather than making a vague request for help.
8. Do the homework, and make sure each answer is correct (check the back of the book, or teacher-provided solutions). As you go through problems you should be gaining new insights and learning what common errors to avoid.
9. Prepare your own study guide for the exam, well ahead of time. It should contain important formulas and facts, and things you learned while doing the problems.
10. Go over your corrected exam. If you didn't learn it before the exam, you can learn it now so you'll know it for the AP exam.

## I. The Idea of Calculus

Calculus deals with change by dividing it up into infinitely small parts, denoted by the letter d .

It was built on the idea that $\mathrm{x}+\mathrm{dx}=\mathrm{x}$. This was later replaced by limits: $\lim _{\Delta \mathrm{x} \rightarrow 0} \mathrm{x}+\Delta \mathrm{x}=\mathrm{x}$.

The infinitely small parts can be summed back up into a whole: $\int d x=x$.

As you work, distinguish between changing quantities and constants.

An infinitely small piece of a curve is just like a straight line.

When change happens at a constant rate calculations are easy. If an object is moving at a constant speed we can find that speed by taking the distance traveled, which is the change in position, and dividing it by the elapsed time to find the speed. Unfortunately, in the real world changes rarely happen at a constant rate. Instead things are usually speeding up or slowing down. In the $17^{\text {th }}$ century, the scientific revolution was in full swing as people sought to explain all kinds of phenomena in the natural world. Regular math was just not meeting their needs. The challenge for mathematics was to come up with a way to do calculations involving "changing" change. Thanks to Isaac Newton and Gottfried Leibniz we now have that way.

We can understand both constant and non-constant change by using calculus. When we are dealing with a changing quantity, a shape, or a curve, we can make meaningful calculations by dividing the item in question up into infinitely many infinitely small parts. Each individual part is infinitely small. During or within that infinitely small part, change is negligible. Surprisingly, we can reach perfectly accurate conclusions this way. Over time this simple idea has become layered with complexity. A quick look at how calculus was originally developed will help you understand the basic ideas, as well as the meaning of the old symbols that are still in use today.


Take some quantity x , which could represent a distance, area, volume, a length of time, etc., and consider an infinitely small change in it. This infinitely tiny change, less than a single grain of sand added to a whole beach, is called dx. The term dx is an abbreviation for "the difference in x ", now officially called the differential of x . The first part of calculus is actually called differential calculus, but because "differential" sounds complicated I will just use the original word "difference" instead. In algebra, when you place two letters next to each other it is assumed that they represent quantities that should be multiplied. That is not the case for dx ; dx is a single entity that is represented by two letters. Anyway, if the quantity x increases by an infinitely tiny amount, its size would be $x+d x$ :


If we had some other quantity $y$, the infinitely small difference of $y$ would be $d y$. If $y$ increases by an infinitely small amount, its size would be $y+d y$ :


Another way of looking at this is that dy is the difference, or the new size minus the old size: $d y=(y+d y)-y$. You should keep in mind that the difference could be negative, since $y$ (or $x$, or any other quantity) could be decreasing instead of increasing. In calculus you may express the difference of anything by putting a d in front of it. If you have some area $A$, then the difference of $A$ is called $d A$, and it represents an infinitely small change in the area. $d V$ would be the infinitely small difference of a volume V , and so on. Not too hard so far.

In algebra, $\mathrm{x}, \mathrm{y}$ and other letters often simply represent fixed unknown quantities. We can solve for them and find one or two values for them that will "fit" an equation. Now that you have reached calculus you will use such letters to represent quantities that can increase or decrease. Isaac Newton called these changing, or "flowing", quantities "fluents". Letters near the end of the alphabet will represent changing quantities. Lowercase letters at the beginning of the alphabet like $a, b$, and especially $c$, are often used for constants. Since the definition of a constant is that it never increases or decreases, the difference of a constant is always zero.

## Example

When Myra checks out her new college calculus textbook, she notices that it contains 4860 different problems. If the number of problems in Myra's textbook is represented by the letter P , find dP .

Sometimes people will rip pages out of a magazine so they can keep a good article or a recipe, but it seems unlikely that anyone would steal a page from Myra's calculus book to obtain a particularly exciting problem. We expect $P$ to stay constant, even when we consider all other possible variables. P will be the same now or at the end of Myra's course, if she keeps or sells her book, or if the book ends up being used for a different course at another college. dP should be the difference, or the "new" value minus the "old" value. $d P=(P+d P)-P$. Since the new
and old values are identical for the constant P , regardless of how we define "new" and "old", dP is zero. $d P=d(4860)=0$. Notice that $d P$ is zero even though $P$ is quite a large number.

As we said earlier, an infinitely tiny change in the quantity x is represented by dx . Implied in this is that we can divide $x$ into infinitely many infinitely small parts, all of size $d x$.

Ideally the only difficulty in calculus should be the term "infinitely small". How small is that exactly? Some people would like to avoid having to imagine this by saying that infinitely small is really zero. I would actually prefer that too so I don't get dizzy thinking about it. However, when we divide a variable quantity like $x$ into infinitely many infinitely small parts, we want to be able to put it back together. That is, the sum of all of those infinitely tiny dx 's must be x . You can't really do that if you say that each dx is zero, so let's stick with " dx is infinitely small". Just like "infinitely large", infinitely small is a concept rather than a specific size. In calculus the sum of the infinitely small parts is abbreviated as $S$, but back when calculus was discovered a special elegant letter $S$ was used to represent this sum: $\int$. So, when we go to put x back together we take the sum of all the $d x$ 's, which is abbreviated as $\int d x$. Now we can say that $\int d x=x$. This just states that $x$ is the sum of the infinitely many, infinitely small $d x$ 's that we have divided it into. In the same way $\int d y=y$ and $\int d A=A$. In calculus, the sum is called an integral (just like an integer is a whole number, an integral is the whole thing).

If you increase something by an infinitely tiny amount, is it really larger than it was before?? The odd idea that $\mathrm{x}+\mathrm{dx}$ is equal to x is what makes calculus possible. We can add differences, multiply them, and even divide by them. Then we say that $\mathrm{x}+\mathrm{dx}=\mathrm{x}$, which makes dx just disappear when it is no longer required. Once the idea was there it was surprisingly easy to set up the framework of this new math. Calculus worked, for constant change as well as changing change, and it provided the wings on which Newton's imagination soared through the solar system. Science advanced quickly as people in Britain and Europe began using this powerful new tool to help them solve problems that had been beyond their reach before.

After the idea of calculus became widely known Bishop Berkeley mocked differences as "ghosts of departed quantities", and called conclusions based on them invalid: "For when it is said, let the Increments vanish, i.e. let the Increments be nothing, or let there be no Increments, the former supposition that the Increments were something, or that there were Increments, is destroyed, and yet a consequence of that supposition, i.e. an expression got by virtue thereof, is retained." Berkeley is saying that differences are either there or not. You can't just use them to draw some conclusion and then have them conveniently disappear. Because he did have a point there, mathematicians eventually brought in the concept of limits to make the basic idea of calculus work in a more elegant way. Instead of the infinitely small change in $x$ that is called
dx , they considered a regular small change in x . "The change in x " was called $\Delta \mathrm{x}$, using the Greek letter $d$, delta. Now imagine that you have $x+\Delta x$, and $\Delta x$ gets smaller and smaller. The limit of this process is that you would eventually just have $x$. A very careful definition of the word "limit" provided a counter-argument to Bishop Berkeley's objection. Your course will devote an entire chapter to limits, and so will this e-book. We do this to avoid, or at least sidestep, the problem of having to say that $\mathrm{x}+\mathrm{dx}=\mathrm{x}$ so that dx must be nothing.

An alternative and completely unofficial explanation of $x+d x=x$ is provided below for your entertainment. Please read it critically and form your own ideas.

Because infinity is a concept rather than a number, it has some very peculiar properties. For example, there are infinitely many counting numbers. However, half of those are odd numbers, and there are also infinitely many of them. The other half are even numbers, and again there are infinitely many of those. It seems that $\infty \div 2=\infty$, and $\infty+\infty=\infty$. If you have infinitely many of something, and you add one more, you still have infinitely many items: $\infty+1=\infty$.

Infinitely small has some of these same peculiar properties, such as "infinitely small" $\div 2$ = "infinitely small". Also, if you actually had something infinitely small, everything else would become infinitely large by comparison. That is quite a scary thought. When Kamex first pointed this out to me I was plagued by an irrational fear of very small objects for some time. Fortunately, there probably is no such thing as "infinitely small" in the real world. The universe appears to have a smallest unit of time, called the Planck time (probably about $1 \times 10^{-43}$ seconds) and a smallest unit of distance, called the Planck length (thought to be about $1.6 \times 10^{-35}$ meters). Mathematics is not restricted by such realities however, and we can imagine that dx exists as a theoretical infinitely small change in the variable $x$. Now consider $x+d x$. By placing an infinitely small increment in $x$ next to the variable $x$, we have actually made $x$ infinitely large by comparison. Most of the time this goes unnoticed and doesn't cause any problems. A picture of an infinitely large object drawn on paper looks the same as a picture of a regular object, and the paper may be safely disposed of at your local recycling facility. However, if you do multiple calculations involving the same infinitely large object, or compare two such objects, you could run into an inconvenient paradox.

Odd as it may seem, $\mathbf{x}+\mathbf{d x}=\mathbf{x}$; that is, adding dx to $\mathbf{x}$ does not change its size even though $d x$ is not equal to zero. If $x$ has temporarily become infinitely large, that may not really be a contradiction. Another way of looking at this is to say that $x$ is a set that contains an infinite number of increments of size $d x . x+d x$ is a set that contains an infinite number of increments of size dx, plus one. Both sets are infinitely large, so the additional dx does not contribute anything to the total size of x .

When we are dealing with infinity, we tend to toss the concept around in a casual manner. One infinity is just like another infinity - they all look the same to us (even though Georg Cantor already proved that there are some infinities that are much larger than others). Fortunately we are more careful with the infinitely small differences we use in calculus. The key seems to be to establish a relationship between variables before we shrink them down to something infinitely small. That is what functions are for Then we name these infinitely small quantities so that we can compare them and distinguish them from each other, even though they are all "infinitely small".

To better understand change, we make graphs of it. Steady change usually graphs as a straight line, while "changing" change ends up looking like a curve of some sort on our graphs. When we use the idea of an infinitely small change along with such curves, it is effectively like zooming in on the curve:


An infinitely small piece of a curve is just like a straight line.
These are the basic ideas of differential and integral calculus. They have been expanded to help people solve many different kinds of problems.

## Reference: Exponents

$$
\begin{array}{ll}
9^{a} \cdot 9^{b}=9^{a+b} & \left(9^{a}\right)^{b}=9^{a b} \\
\frac{9^{a}}{9^{b}}=9^{a-b} & (9 e)^{a}=9^{a} e^{a} \\
9^{-a} \text { is } \frac{1}{9^{a}} & 9^{\frac{1}{n}}=\sqrt[n]{9} \\
9^{0}=1 & 9^{\frac{a}{n}}=\sqrt[n]{9^{a}} \text { or }(\sqrt[n]{9})^{a}
\end{array}
$$

Exponents are very important in calculus, and they are also the second item in "Please Excuse My Dear Aunt Sally". This means that they have a very high priority in the order of mathematical operations. An exponent on something generally belongs exclusively to that one thing, unless there are parentheses present. So, $9 \mathrm{e}^{2}$ is just $9 \mathrm{e}^{2}$, because the exponent applies only to the e. However, once you place some parentheses they get priority, and the exponent applies to the entire thing inside those parentheses. $(9 \mathrm{e})^{2}$ means $9 \mathrm{e} \cdot 9 \mathrm{e}$ which is $81 \mathrm{e}^{2}$.

First we will do some multiplication involving exponents, which is the easiest part. $9^{3} \cdot 9^{4}=$ ? This really means $9 \cdot 9 \cdot 9$ times $9 \cdot 9 \cdot 9 \cdot 9$. Now you have seven $9^{\prime}$ s in a row, which is $9^{7}$. For multiplication, just add the exponents.
$\frac{9^{5}}{9^{3}}=\frac{9 \cdot 9 \cdot 9 \cdot 9 \cdot 9}{9 \cdot 9 \cdot 9}=9^{2}$. For division, we can subtract the exponents. Notice that by subtracting exponents, we find that $\frac{9^{1}}{9^{1}}$ is $9^{1-1}$ which is $9^{0}$. Since any number divided by itself is $1,9^{0}$ must mean 1. That works for every other number too, except for zero. Since we can't divide by zero, $0^{0}$ is really undefined. You might think that $0^{0}$ should just be 1 too, but on the other hand 0 to any other power is 0 . When something is not defined, you can often get a handle on it by sneaking up on it: $0.25^{0.25}=0.70710 \ldots, 0.1^{0.1}=0.79432 \ldots, 0.01^{0.01}=0.95499 \ldots$. The limit of this process is 1 . If you really need to have a value for $0^{0}$, you should probably use 1.

Another thing that happens when you start subtracting exponents is that you end up with negative exponents. Consider $\frac{9^{2}}{9^{5}}$. You can write that as $\frac{9 \cdot 9}{9 \cdot 9 \cdot 9 \cdot 9 \cdot 9^{\prime}}$, and cross off two $9^{\prime}$ 's on the top and bottom. Be careful because there is still a 1 on the top after you are done with
that: $\frac{9 \cdot 9 \cdot 1}{9 \cdot 9 \cdot 9 \cdot 9 \cdot 9}=\frac{1}{9^{3}}$. If you had subtracted the exponents to begin with that would leave you with $9^{-3}$. This tells you that $9^{-3}$ means $\frac{1}{9^{3}}$.

You can also put an exponent on something that already has an exponent. This looks like $\left(9^{2}\right)^{3}$. Thinking carefully about what that means, we rewrite it as $9^{2} \cdot 9^{2} \cdot 9^{2}=9^{6}$. So, when you raise a power to another power, you multiply the exponents. Be careful when there are multiple things inside the parentheses. (9e) ${ }^{3}$ means $9 \mathrm{e} \cdot 9 \mathrm{e} \cdot 9 \mathrm{e}$, or $9^{3} \mathrm{e}^{3}$.

What about exponents that are fractions? Let's consider $9^{\frac{1}{2}}$. We know that when we multiply things with exponents, we can just add the exponents. So $9^{\frac{1}{2}} \cdot 9^{\frac{1}{2}}=9^{\frac{1}{2}+\frac{1}{2}}=9^{1}$. This means that $9^{\frac{1}{2}}$ has to represent a number that, when multiplied by itself, is 9 . The only candidate for this is $\sqrt{9}$, since $\sqrt{9} \cdot \sqrt{9}=9$. So what is $9^{\frac{1}{3}}$ ? There would have to be a number such that $9^{\frac{1}{3}} \cdot 9^{\frac{1}{3}} \cdot 9^{\frac{1}{3}}=9$. This is the number we call the cube root of 9 , or $\sqrt[3]{9}$. By now you can probably guess that $9^{\frac{1}{4}}$ is $\sqrt[4]{9}$, and so on.

Calculus also involves more complicated fractional exponents like $9^{\frac{3}{2}}$. Just use your knowledge of fractions to see how such an exponent could have been created. We know that when we raise a power to a power, the two numbers are multiplied. That means that $9^{\frac{3}{2}}$ could have been created in two ways: $\left(9^{3}\right)^{\frac{1}{2}}$ or $\left(9^{\frac{1}{2}}\right)^{3}$. Therefore, $9^{\frac{3}{2}}=\sqrt{\left(9^{3}\right)}=(\sqrt{9})^{3}$. Both these forms mean the same thing. Notice that it is the denominator of the fractional exponent that determines what kind of root is involved. Scary-looking fractional exponents follow the same rules as regular exponents. Add them when you multiply, and subtract them when you divide.

## Reference: Useful Trigonometry

While you may not think of trigonometry as being useful in any way, it is scattered all through your calculus course. A few helpful trigonometry facts can make your life a lot easier. Use this rather boring reference section when you need it, and skip ahead to the next chapter for now. For additional trigonometry help see: Algebra 2 \& Pre-Calculus: "Trigonometry".

| Angle (radians) | Angle (degrees) | Sine | Cosine |
| :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 1 |
| $\pi / 6$ | 30 | $1 / 2$ | $\sqrt{3} / 2$ |
| $\pi / 4$ | 45 | $\sqrt{2} / 2$ | $\sqrt{2} / 2$ |
| $\pi / 3$ | 60 | $\sqrt{3} / 2$ | $1 / 2$ |
| $\pi / 2$ | 90 | 1 | 0 |

In quadrant I the sine gets larger as the angle gets larger, and the cosine gets smaller. You should either have these values memorized, or preferably be able to derive them by drawing a unit circle.
$(\sin (x))^{2}$ is normally written as $\sin ^{2} x$. By the Pythagorean Theorem,

$$
\sin ^{2} x+\cos ^{2} x=1
$$



Divide both sides of this equation by either $\sin ^{2} x$ or $\cos ^{2} x$ to get two more useful identities that you won't have to memorize.

```
\mp@subsup{\operatorname{sin}}{}{2}x+\mp@subsup{\operatorname{cos}}{}{2}x=1
\mp@subsup{\operatorname{tan}}{}{2}x+1=\mp@subsup{\operatorname{sec}}{}{2}x
1+ \mp@subsup{\operatorname{cot}}{}{2}x=\mp@subsup{\operatorname{csc}}{}{2}x
```

Unfortunately, if you need the sine or cosine of two angles added together, you can't find the sine and cosine separately and then add them up. $\operatorname{Sin}(x+y)$ is not equal to $\sin x+\sin y$ ! Instead:
$\begin{array}{ll}\sin (x+y)=\sin (x) \cos (y)+\cos (x) \sin (y) & \sin (x-y)=\sin (x) \cos (y)-\cos (x) \sin (y) \\ \cos (x+y)=\cos (x) \cos (y)-\sin (x) \sin (y) & \cos (x-y)=\cos (x) \cos (y)+\sin (x) \sin (y)\end{array}$

When $x=y$ we get two very useful equations that you will be expected to know:

```
sin}(2x)=2\operatorname{sin}(x)\operatorname{cos}(x
```



The second formula can be used to simplify problems that contain the square of the sine or cosine:
$\cos (2 x)=\cos ^{2} x-\sin ^{2} x$
$\begin{array}{lll}\cos (2 x)=\left(1-\sin ^{2} x\right)-\sin ^{2} x & \text { or } & \cos (2 x)=\cos ^{2} x-\left(1-\cos ^{2} x\right) . \\ \cos (2 x)=1-2 \sin ^{2} x & & \cos (2 x)=2 \cos ^{2} x-1 \\ 2 \sin ^{2} x=1-\cos (2 x) & & 2 \cos ^{2} x=1+\cos (2 x)\end{array}$
Therefore:

$$
\sin ^{2} x=\frac{1}{2}(1-\cos 2 x) \quad \cos ^{2} x=\frac{1}{2}(1+\cos 2 x)
$$

## II. Limits and Continuity

Knowing how to factor is important in this section. Remember that $a^{2}-b^{2}=(a+b)(a-b)$.

Continuous functions have no holes or breaks in their graphs.

An infinitely small hole (removable discontinuity) appears when a rational function has a factor in the numerator that cancels out a factor in the denominator. The limit exists at the hole.

In order for a limit to exist, the left-hand limit must equal the right-hand limit.

Limit $=\infty$ means that the limit doesn't exist.

To survive your chapter on limits you must know that
$a^{2}-b^{2}=(a+b)(a-b)$
If you want to see how this works, and remember it better, grab a piece of paper and some scissors. Cut yourself a square with sides $a$, and then remove a square with sides $b$ from it like this:


This represents $a^{2}-b^{2}$. Next, cut along the dotted line and rearrange:


Now you can see that $\mathrm{a}^{2}-\mathrm{b}^{2}=$ length $\cdot$ width $=(a+b)(a-b)$.
An example of this would be $x^{2}-9=(x+3)(x-3)$. Don't forget that 1 is a square too. $y^{2}-1=$ $(y+1)(y-1)$.

Either a or b, or sometimes both, could have exponents:
$a^{4}-25=\left(a^{2}+5\right)\left(a^{2}-5\right)$
Problems may also use square roots. Just as $x^{2}-9=(x+\sqrt{9})(x-\sqrt{9})$, you can write $x^{2}-$ a as $(x+\sqrt{a})(x-\sqrt{a})$.

This chapter in your textbook introduces the idea of limits. To ease you into the topic your book will ask you to use these limits on something that you are already familiar with: finding the value of a function. In the picture below, you can see the graph of a function.


At point $A$, the value of $x$ is 1 . When you put that into the equation of the function, $y=-x^{3}+2 x+2$, you get a $y$-value of 3 . We say that the value of the function is 3 when $x$ is equal to 1 . If $x$ is not 1 , but just a little bit bigger, you can say that the value of the function is close to 3 . The closer $x$ gets to 1 , the closer the function value gets to 3 . In this case we would say that x is approaching 1 from the right, since we are considering values of x that are to the right of 1 on the $x$-axis. We could also look at $x$ values smaller than 1 , and then get closer and closer to 1 from the left. Either way, the function value approaches 3 as x approaches 1 more and more closely. We say that the limit of the function value is 3 as $x$ approaches 1 . By the way, I will use $y=-x^{3}+2 x+2$ interchangeably with $f(x)=-x^{3}+2 x+2$. The " $y=. .$. " notation looks simpler and it is what you actually use to construct a graph of the function. The " $\mathrm{f}(\mathrm{x})=\ldots$..." way of writing a function allows us to specify what value of $x$ was used to obtain a particular value of $y$ shown on the graph: $f(1)=3$ is a quick way to write that $y$ is equal to 3 when $x$ is equal to 1 .

Limits are very straightforward when the function has a continuous graph. However, some functions have a break or a hole in the graph. For example, $y=\frac{x^{2}-9}{x-3}$ does not have a value when $x=3$. The best we can do is get really, really close to 3 and see what happens. You may want to take your calculator and try that out. You should find that y gets closer and closer to 6 as $x$ gets closer and closer to 3 . This happens whether you start at 2.9 and go up, or at 3.1 and go down. Even though the function does not have a value at $x=3$, we say that the limit of the function value, as $x$ approaches 3 , is 6 .

As you do your calculations, you might notice that when you get really close to 3, the bottom part of the fraction gets to be very small. Normally, when you divide something by a very small decimal number, like .0001, you end up with a large result. In this case however, the top part of the fraction also gets smaller. Just calculate the top and the bottom parts separately to see that. The reason is that $x^{2}-9$ is the same as $(x-3)$ times $(x+3)$. The bottom part of the fraction can get extremely small, but it is always cancelled out by the $(x-3)$ in the top part. In the end, the actual value of the fraction is always what is left, $x+3$. Even though $x$ is not permitted to be 3 , the value of the function, $x+3$, approaches 6 as $x$ approaches 3 .

Let's see how this kind of thing was done before anyone had even heard of calculators or limits. Here is an example from the first calculus textbook, published in 1696:

## Example

$y=\frac{a^{2}-a x}{a-\sqrt{a x}}$, where $a$ is a positive constant. Find the value of $y$ when $x=a$.
Here $x$ and $y$ represent variable quantities, while $a$ is a constant. When $x$ is equal to $a$, both the numerator and the denominator of the fraction will be zero. Notice that there is no mention of the word limit at all at this early date, or any major concern about potentially dividing by zero. This sample problem was solved by eliminating the nasty square root on the bottom (although if you look closely you may be able to do it faster by spotting the difference of two squares).
$y=\frac{a^{2}-a x}{a-\sqrt{a x}} \cdot \frac{a+\sqrt{a x}}{a+\sqrt{a x}}$ Don't multiply out the top part; we are looking for stuff to cancel out: $y=\frac{\left(a^{2}-a x\right)(a+\sqrt{a x})}{a^{2}-a x}$
$y=a+\sqrt{a x}$ Now let $x=a$ :
$y=a+\sqrt{a^{2}}$
$y=2 a$
Wow, this was truly the Wild West of mathematics. They divided by $\mathrm{a}^{2}-\mathrm{ax}$ first, and then made it equal to 0 . That looks like they were cheating and dividing by zero! Now that things are more civilized ... we just do the same thing but we cover it up much better.; Rather than saying that $x$ is equal to $a$, we make sure that this never officially happens. Instead, we use the limit as $x$ approaches a. Once the division has been accomplished, it becomes obvious that the limiting value of this limit process would be $2 a$. We don't actually claim that $y=2 a ;$ we just say that the limit of $y$, as $x$ approaches $a$, is $2 a$.
$\lim _{x \rightarrow a} \frac{\left(a^{2}-a x\right)(a+\sqrt{a x})}{a^{2}-a x}=\lim _{x \rightarrow a}=a+\sqrt{a x}=a+\sqrt{a^{2}}=2 a$
Division by zero? It never happened, or at least you can't prove that it did. Problem solved.
You should become comfortable with the idea that a limit is something that you may not be able to reach but that you can come as close to as you want. To make the definition of the word limit even more solid, textbook examples will feature piecewise defined functions. The image below shows the piecewise function $y=\frac{x^{2}-9}{x-3}$ if $x \neq 3$ and $y=4$ if $x=3$ :


The limit of $y$ is still 6 as $x$ approaches 3 , because that is what the function value keeps getting closer and closer to. The actual value of the function is 4 when $x$ is equal to 3 , so nobody can claim that we are just using this limit thing to justify a division by zero.

Caution: the open circle shown on the graph at the point $(3,6)$ is misleading in the sense that it is a lot smaller than it looks. The "hole" that it represents is infinitely small so you shouldn't be able to see it at all. In fact, $x$ could be 3.0000000000000000000000000000000000000001 without causing any problems. Realizing this may make it easier for you to understand that there is a still a limit at such a point.

Although the hole is incredibly small, it is there nevertheless. The function is not continuous. These infinitely small holes appear whenever a rational function has a factor in the numerator
that cancels out a factor in the denominator. For a function like $f(x)=\frac{x^{2}-4}{x-2}, x \neq 2$ the discontinuity at $x=2$ can be removed by actually doing the division and replacing $f(x)$ with the new function $\mathrm{g}(\mathrm{x})=\mathrm{x}+2$ which has no restriction on x . A discontinuity involving a hole like this is called a removable discontinuity.

As you go through your chapter you will learn about left-hand limits and right-hand limits. Again, these limits refer to function values. To get a right-hand limit, put your finger on the function graph to the right of the point at which you are trying to find the limit, and then move closer to it. For the left-hand limit you start to the left and move right. If you end up with two different values when you do this, the Limit does not exist and you will not be able to use calculus to find how fast the function is changing at that point. This can happen if there is a break or gap in the function.

If there is no limit for the function value, calculus will not be useful, at least at that particular point. Another case where this happens is if there is no limit because the function values keep getting larger and larger or smaller and smaller near the point in question. We say that the limit is positive or negative infinity, but that really means it does not exist. It is like saying, "The sky is the limit." This expression means that there is no limit.
$\operatorname{Lim}=\infty$ means there is no limit.

This kind of thing is usually caused by a rational function that doesn't have a factor in the top part that cancels out the division on the bottom. For example, if $f(x)=\frac{x^{2}+4}{(x-2)^{2}}$, a value of $x$ very close to 2 causes a division by a very tiny number, and there is nothing in the numerator that cancels it out. That makes the function value shoot to infinity as $x$ gets close to 2 , creating a vertical asymptote at $x=2$. An asymptote is a line that the graph of a function approaches more and more closely, but never reaches. The function $f(x)=\frac{x^{2}+4}{(x-2)^{2}}$ is shown below, along with the asymptote at $x=2$. The limit, as $x$ approaches 2 , is infinity. The limit does not exist.


Looking for $\lim x \rightarrow a$ of a rational algebraic (no trig, logs or $b^{x}$ ) function? There are 3 possibilities:

1. The function exists at the point in question so you can plug in the value directly
2. The top and bottom have a common factor (there is a removable discontinuity)
3. There is no common factor so the function has a vertical asymptote ( $\lim = \pm \infty$ ).

Don't hesitate to use your graphing calculator or graphing software to see what is going on.

Limits can often be computed by using algebraic manipulations. Usually these manipulations involve: factoring, the difference of two squares, the difference or sum of two cubes, and the difference of two squares in reverse. That last part is useful when you see a fractional
expression that has $\sqrt{\ldots}-\sqrt{\ldots}$ on the top or the bottom. Multiply by $\frac{\sqrt{\ldots}+\sqrt{\ldots}}{\sqrt{\ldots}+\sqrt{\ldots}}$ to eliminate the radicals on one end, and don't worry about the other end. Something will probably cancel out, allowing you to find the limit.

As you rearrange limit expressions using various tricks, do not rush to multiply out the terms on the top or the bottom. Since you are looking for things to cancel out, the factored form is usually better.

If there is no way to change the expression in your problem to a more favorable one, you may be expected to find the limit simply by considering what actually happens when $x$ approaches the indicated value.

The next example involves the natural logarithm function, In (x). You may recall that this function expresses any number $x$ as the number e raised to some power. Below is a picture of $y=\ln x:$


## Example

Find $\lim _{x \rightarrow 0^{+}} \frac{1-\ln x}{x}$
[Survival Tip: Don't just mindlessly read ahead to get the answer to examples in a text. Cover up the solution and think about the problem first. Even if you can't solve it, you will be better prepared to understand and remember the explanations provided in your book.]

So what's with that $\mathrm{x} \rightarrow 0^{+}$part? As a student, I learned to ignore that kind of thing and just look for the limit as $x$ goes to 0 . But now that I have to make my own problems I need to pay more attention. I can't just ask for $\lim \mathrm{x} \rightarrow 0$ here because $\ln \mathrm{x}$ doesn't exist to the left of $\mathrm{x}=0$. The limit must be approached from the right only. Since you'll probably already do that intuitively it won't matter much to you. Consider what happens to $\ln x$ as $x$ gets very small. You can see that the graph takes a nosedive when it gets near zero. The reason for that is that the only way to create a very small number using $\mathrm{e}^{\mathrm{x}}$ is to put a large negative exponent on e . The natural log function gives you that exponent, so it returns an extremely large negative number when x is near 0 . Try it out on your calculator. This means that $1-\ln \mathrm{x}$ will be an extremely large positive number when x is near 0 . And what happens when you take a really large number and divide it by a really tiny number? It just gets even bigger. $\lim _{x \rightarrow 0^{+}} \frac{1-\ln x}{x}=\infty$, which means that it doesn't exist. You can check your work by entering $y=\frac{1-\ln x}{x}$ in your favorite graphing app.

## The Limit Laws

Limits obey the following rules. If $f$ and $g$ are functions and a and c are constants, then:

1. $\lim _{\mathrm{x} \rightarrow \mathrm{a}} f+\mathrm{g}=\lim _{\mathrm{x} \rightarrow \mathrm{a}} f+\lim _{\mathrm{x} \rightarrow \mathrm{a}} g$
2. $\lim _{\mathrm{x} \rightarrow \mathrm{a}} f-g=\lim _{\mathrm{x} \rightarrow \mathrm{a}} f-\lim _{\mathrm{x} \rightarrow \mathrm{a}} g$
3. $\lim _{\mathrm{x} \rightarrow \mathrm{a}} \mathrm{c} f=\mathrm{c} \lim _{\mathrm{x} \rightarrow \mathrm{a}} f$
4. $\lim _{\mathrm{x} \rightarrow \mathrm{a}} f \cdot g=\lim _{\mathrm{x} \rightarrow \mathrm{a}} f \cdot \lim _{\mathrm{x} \rightarrow \mathrm{a}} g$
5. $\lim _{x \rightarrow a} \frac{f}{g}=\frac{\lim _{x \rightarrow a} f}{\lim _{x \rightarrow a} g}$ provided $g \neq 0$
6. $\lim _{x \rightarrow a} f^{n}=\left(\lim _{x \rightarrow a} f\right)^{n}$
7. $\lim _{\mathrm{x} \rightarrow \mathrm{a}} \sqrt[n]{f}=\sqrt[n]{\lim _{\mathrm{x} \rightarrow \mathrm{a}} f}$

There are also some basic common sense rules. For example, the limit of 5 is 5 , and if $x$ approaches $a$, the limit of 5 is still 5 . $\lim _{x \rightarrow a} c=c$ where $c$ is a constant. The limit of $x$, as $x$ approaches $a$, is of course $a: \lim _{x \rightarrow a} x=a$. That last statement leads to these next two:
$\lim _{x \rightarrow a} x^{n}=a^{n}$
$\lim _{x \rightarrow a} \sqrt[n]{x}=\sqrt[n]{a}$
This says that if, for example, $x$ approaches 4 and $f(x)=\sqrt{x}$ then the limit of that is $\sqrt{4}=2$.

## Limits at Infinity

To take advantage of the limit of $\frac{1}{x^{\prime}}$, divide the top and bottom polynomial by the highest power of $x$ in the denominator.

1. If the highest power of $x$ on the top is smaller than the highest power of $x$ on the bottom, the function will always approach 0 as $x \rightarrow \infty$.
2. If the leading term on the top has the highest power, the function will approach positive or negative infinity.
3. If the highest powers of $x$ on the top and bottom are equal, the function approaches a definite number which is determined by the coefficients of the leading terms.

Sometimes you will be asked to find the limit of a function $f(x)$ as $x$ goes to infinity. If the function in question is composed of one polynomial divided by another polynomial, there is a handy trick we can always use. This trick is based on the fact that the limit of $\frac{1}{x}$ is zero as $x$
becomes infinitely large. Let's see how it works.

To take advantage of the limit of $\frac{1}{x^{\prime}}$, divide the top and bottom polynomial by the highest power of $x$ in the denominator. Note that this means dividing every single term, because it is easy to forget some.

## Example

Find the limit as $x \rightarrow \infty$ of $f(x)=\frac{x^{3}+3 x^{2}+4 x+7}{x^{5}+x^{4}+2 x^{2}+10}$.
What happens to this function when we have very large or very small values of $x$ ? To find out, divide all the terms by the largest power of $x$ in the denominator, which is $x^{5}$ in this case. After simplifying, we end up with $f(x)=\frac{\frac{1}{x^{2}}+\frac{3}{x^{3}}+\frac{4}{x^{4}}+\frac{7}{x^{5}}}{1+\frac{1}{x}+\frac{2}{x^{3}}+\frac{10}{x^{5}}}$. Now we have to stop and consider what happens to something like $\frac{10}{\mathrm{x}}$ when x gets very large (or very negative). If you think about that for a while you will realize that it is just the same as for $\frac{1}{\mathrm{x}}$; the expression eventually gets very close to zero. If we replace all of the expressions of the form $\frac{a}{x^{n}}$ with zero, we see that the value of this function approaches 0 as $x$ approaches positive or negative infinity. That means there will be a horizontal asymptote at $\mathrm{y}=0$.

## Example

Find the limit as $x \rightarrow \infty$ of $f(x)=\frac{x^{2}}{\sqrt{x^{3}+2 x^{2}}}$.
Be careful, because the highest power of $x$ is really $x^{3 / 2}$ since $x^{3}$ is underneath a square root sign. [Remember that $\sqrt{x}$ is the same as $x^{1 / 2}$, so $\sqrt{x^{3}}$ is $\left(x^{3}\right)^{\frac{1}{2}}$.] When you divide $x^{2}$ on the top by $x^{3 / 2}$ you subtract the exponents to get $x^{1 / 2}$. On the bottom of the fraction we can divide by $x^{3}$ because we are working under the square root sign:
$\lim _{x \rightarrow \infty} \frac{x^{1 / 2}}{\sqrt{\frac{x^{3}}{x^{3}}+\frac{x^{2}}{x^{3}}}}$
$\lim _{x \rightarrow \infty} \frac{\sqrt{x}}{\sqrt{1+\frac{2}{x}}}$
The limit of $\frac{2}{x}$ as $x$ goes to infinity is zero, which leaves $\sqrt{1}=1$ on the bottom. As $x$ goes to infinity so does the square root of $x$ on top, which means that the limit is infinity.

Your textbook will likely have many examples for you to try. After you have been doing these problems for a while, you will notice a pattern:

1. If the highest power of $x$ on the top is smaller than the highest power of $x$ on the bottom, the function will always approach 0 as $\mathrm{x} \rightarrow \infty$.
2. If the leading term on the top has the highest power, the function will approach positive or negative infinity.
3. If the highest powers of $x$ on the top and bottom are equal, the function approaches a definite number which is determined by the coefficients of the leading terms.

## Trigonometric Limits

$$
\lim _{x \rightarrow 0} \frac{\sin x}{x}=1
$$

Since there are not really that many different types of limit problems, your course may use a special trigonometry limit as well. Consider $y=\frac{\sin x}{x}$. Zero is not in the domain of this function, but as it turns out $\lim _{x \rightarrow 0} \frac{\sin x}{x}$ is quite important because it is used to show a special relationship
between the sine and the cosine. The graph of the function $y=\frac{\sin x}{x}$ is shown below, along with $y=\sin x$ (the red line). You can see that the limit of $\frac{\sin x}{x}$ is 1 as $x$ approaches zero:


There is just an infinitely tiny hole at $x=0$ that you can't see in the picture. You can also create a table of function values for smaller and smaller values of $x$ to see that the function approaches 1 as you get closer and closer to $x=0$. Introducing the special $\operatorname{limit}_{x \rightarrow 0} \frac{\sin x}{x}=1$ at this point opens up the possibility of making you review your trigonometry. You may be asked to rearrange various expressions so you can take advantage of this limit to solve other trigonometric limits.

## Example

Find $\lim _{x \rightarrow 0} \frac{\tan x}{x}$.
Hopefully you will remember that the tangent is the sine divided by the cosine:
$\lim _{x \rightarrow 0} \frac{\tan x}{x}=\lim _{x \rightarrow 0} \frac{\frac{\sin x}{\cos x}}{x}=\lim _{x \rightarrow 0} \frac{\sin x}{\cos x} \div x=\lim _{x \rightarrow 0} \frac{\sin x}{\cos x} \cdot \frac{1}{x}=\lim _{x \rightarrow 0} \frac{\sin x}{x \cos x}$
Because you know the limit of $\frac{\sin x}{x}$ as $x$ goes to zero, you can solve this by rearranging:
$\lim _{x \rightarrow 0} \frac{\sin x}{x \cos x}=\lim _{x \rightarrow 0}\left(\frac{\sin x}{x} \cdot \frac{1}{\cos x}\right)$
The limit of a product is the product of the limits. For $\lim _{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x}$, write $\lim _{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim _{x \rightarrow 0} \frac{1}{\cos x}$

Now you can substitute the values: $1 \cdot \frac{1}{\cos 0}=1 \cdot \frac{1}{1}=1$.
So, $\lim _{x \rightarrow 0} \frac{\tan x}{x}=1$

## Average Rate of Change

The average rate of change is $\frac{y_{2}-y_{1}}{x_{2}-x_{1}}$, which is the same as $\frac{f\left(x_{2}\right)-f\left(x_{1}\right)}{x_{2}-x_{1}}$

Before you get to derivatives, your calculus course will probably prepare you by having you consider an average rate of change. A simple way to understand this is by looking at speed. Speed measures the change in position of an object over time. If a car starts at point $A$, and 2 hours later it is at point B which is 130 miles away from point $A$, we say that the average speed of the car is 65 miles per hour. It doesn't matter if the car is sometimes going faster or slower than 65 miles per hour, or even stopped at a light. All we are interested in here is the average speed. That means all you need is the starting position and time, and the end position and time. Then you divide the net change in position by the elapsed time.

You can do the same thing with a function. If the $y$-value of a function is 10 at $x=0$, and 25 at $x=3$, then the net change in the function over the interval is $25-10$ or 15 . Divide that by the length of the interval, $3-0$ or 3 , to get an average rate of change of 5 .

You may see questions about the average rate of change on an exam. Remember that the answer is usually easy to find. $)$

## III. Understanding Change: The Derivative

The Derivative measures the instantaneous rate of change, by taking an infinitely small change in the $y$-value (dy) and dividing it by the infinitely small change in the $x$-value ( $d x$ ).

The Derivative is the slope of the tangent line. It is positive when the function is increasing, and negative when the function is decreasing.


Isaac Newton


Gottfried Leibniz

Non-constant change was nearly impossible for scientists to deal with until Isaac Newton used calculus to determine the speed of objects that accelerate or decelerate. This was a major breakthrough, which was also achieved independently by German philosopher and mathematician Gottfried Leibniz a little later. A controversy over who discovered calculus first, Newton or Leibniz, quickly became a matter of national pride as educated people throughout Britain and Europe realized the importance of this new method. Calculus applies not only to speed, but to all kinds of change.
(Most of this section has been written by Kamex, whose enthusiasm for the subject I can't quite match.)

Speed is defined as how far an object travels in an amount of time (like miles per hour). To find it, take two points and divide the distance by the time that it takes the object to get from point A to point B. But an object's speed isn't always the same; it can change. Things stop, start, slow down, and speed up, so here is the big question: what is an object's speed at a certain point in time?

Imagine you had the power to stop time. When time is stopped nothing moves. That's because you're looking at a single instance in time, and in a single instance of time nothing moves. A period of time is made up of a bunch of instances, so when you put these instances together, and in each instance nothing moves, then how is the object moving at all? When Zeno, a Greek philosopher, first thought up this paradox, no one was able to explain it. But now, with our new technology...we STILL can't explain it.

Not to worry though, there's a way around this little difficulty. It's called the derivative. Most things in our world don't move at a constant speed, and the derivative helps us find the speed of an object at any given point in time.

The first thing we need is an accurate description of the object's motion. This is where all that stuff about functions comes in. If you look at the graph below, you can see that the distance at each point is twice the value of the time, or $\mathrm{D}=2 \mathrm{t}$.


We can use the function $f(t)=2 t$ to describe the motion.
Let's figure out the speed that the object in the graph is traveling at. We can tell that the speed is 2 units of distance per 1 unit of time. We know this because we can take two points on the graph that are one unit of time apart: point A, and point B. The distance that the object travels between these two points in time is 2 units of distance. Because the speed on this graph is all the same, pick a point, and the speed at that point will be 2 . That's simple enough, but it gets more complicated than that. Check out this graph:


Hmm...okay. In this graph, the speed is changing at certain points. This means that there isn't a single speed that the object is traveling at the whole time. If we want to know the speed at some point, we have to just calculate the slope of the line at that particular point. But what about THIS:


Time

Ahhhh!!! In this graph, there is no way to measure the speed because it is constantly increasing. This is because it's a curved line, and we can't find the slope of a curved line, or...can we???

First, let's use a function to describe what is going on here. In this case, the distance is the square of the time, meaning that if $x$ is $3, y$ is $3^{2}$. If $x$ is $12, y$ is 144 . The function is $f(t)=t^{2}$, which you can also write as $D=t^{2}$. We can graph it just like we graph $y=x^{2}$. To calculate the speed along any section of the graph we need to find the slope just like we did before. But any section we look at has a curved line, so it can't be measured. Now how can we find the slope??

If you put two dots on the line, there is a curve between them, but if the two dots are closer together, there is less of a curve between them. The closer together you put them, the more accurate your measurement of the slope is going to be if you treat the middle part as a straight line, because the closer the points get, the less curvy the middle line is. What if we make the two points infinitely close together, meaning that there is an infinitely small distance between them. Wouldn't that make the measurement infinitely accurate?

The slope of a line is "rise over run". The rise is the y change and the run is the x change. (Remember the slope formula: $m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}$ ). Here y is the distance D , and x is the time t , so the slope is the change in $D$ divided by the change in $t$. If the two points are infinitely close together, then the slope would be an infinitely small change in D divided by an infinitely small change in $t$. Hey, wait! That sounds familiar

All we need to find the slope (the speed at a point in time) is the infinitely small difference of $D$, which is dD , divided by the infinitely small difference of t , which is dt . So we are looking for $\frac{\mathrm{dD}}{\mathrm{dt}}$.


Notice that now that we have zoomed in infinitely far, the curve $D=t^{2}$ looks like a straight line. Of course you should keep in mind that this is just a diagram, and nobody has ever seen what things really look like this close up. What we do know is that when we calculate the slope, it will be accurate.

Because we have a function that tells us how $D$ and $t$ are related, it is easy to find $d D$. $D=t^{2}$, and the Power Rule says that $d D=2 t d t$.

We need $\frac{\mathrm{dD}}{\mathrm{dt}}$ so we want to divide both sides by dt . Time increases at a nice steady rate, and it doesn't dependent on any other variable. Each infinitely small bit dt can be considered to be equal in size so that $\mathrm{dt} / \mathrm{dt}=1$ :
$\frac{\mathrm{dD}}{\mathrm{dt}}=\frac{2 \mathrm{tdt}}{\mathrm{dt}}$
$\frac{\mathrm{dD}}{\mathrm{dt}}=2 \mathrm{t}$.
That's nice, but what does it mean? Well, if $\frac{\mathrm{dD}}{\mathrm{dt}}=2 \mathrm{t}$ then at any point in time the speed of the object whose position is shown in the graph is equal to $2 t$. If $t$ is measured in hours, and $D$ is measured in miles, then at time $t=2$ hours the speed would be 2 t or 4 miles per hour. At time
$t=3$ hours the speed would be 6 miles per hour, and so on. Now we know how to find the instantaneous speed of an object that is speeding up or slowing down.

What is also interesting here is that the two points we used to find the slope are so infinitely close together that they are really the same point. That means that the line between them has now become a tangent line, which is a line that just touches a curve at one point. Without calculus it is much more difficult to create such a line, and when Newton's teacher Isaac Barrow first used calculus methods to find a tangent to a curve it was considered a great achievement.


The infinitely small difference in one variable divided by the infinitely small difference in a related variable, like $\frac{\mathrm{dD}}{\mathrm{dt}}$, is called a derivative. The derivative is really amazing if you think about it carefully. If $A$ and $B$ are really the same point, just where is that triangle with the curve, dD, and dt??


The derivative represents how fast the top variable is changing compared to the change in the bottom variable. In the case we just looked at, the derivative represents the rate of change of the distance relative to the change in time. A more old-fashioned way of saying the same thing is that we look at the change in distance with respect to time. A change in distance with time is what we call speed. In this case the derivative represents speed. The slope (rise over run) of the tangent line drawn to the curve represents how fast the position is changing at that point.

Caution: The derivative is the slope of the tangent line, not the line itself!

Derivatives can be used to work with any kind of change. They are useful in physics, engineering, theoretical chemistry, finance, software development, and many other applications. Unfortunately these fields are now so specialized that the average person has difficulty appreciating the contributions that calculus has made, and is continuing to make, to science and technology.

You will be using the derivative with functions. What you will be asked to find is how much the $y$-value of a function is changing relative to the change in $x$, that is, you'll be looking for $\frac{d y}{d x}$. Even when the function graph is a curve, you'll be able to determine this rate of change by using calculus.

The $x$-value on a graph increases from left to right. If you put your pencil on a point on a graph, and move to the right, the function is increasing if the graph is going up (the $y$-value is
increasing), and decreasing if the graph is going down. Because the derivative of a function measures the rate of change of the $y$-value, it is positive when the function is increasing, and negative when the function is decreasing.

## $\Delta$ Notation for the Derivative

$$
\text { The derivative is a limit: } \frac{\mathrm{dy}}{\mathrm{dx}}=\lim _{\Delta \mathrm{x} \rightarrow 0} \frac{\Delta \mathrm{y}}{\Delta \mathrm{x}}
$$

We can find many derivatives through differences, but today all of this work is done by using limits. This explanation allows you to see how the derivative works, how it may change depending on the particular point you select, and how limits fit into this.

In the previous section we saw how to determine how fast the function $D=t^{2}$ is increasing at any given point. Originally calculus was applied to motion, so people used the derivative to find how the speed of an object was changing over time. However, calculus can be applied to any kind of change. For any function $f(x)$, we can measure how $f(x)$ changes as $x$ changes. In the case of $f(x)=x^{2}$, the value of the function increases as $x$ increases. When $x$ is small $f(x)$ increases slowly with each bit of increase in $x$, and as $x$ gets larger the function value increases faster and faster with an increase in x .

To find how fast the function is changing, we can draw the graph of $y=x^{2}$ and place two points on it, point A and point B, connected by a line. The slope of this line is the difference between the $y$ coordinates of the points divided by the difference in the $x$ coordinates. That is, the slope is the change in $y$, called $\Delta y$, divided by the change in $x$, or $\Delta x$. Note that $\Delta y$ is a change in $y$, while dy refers to an infinitely small change in y . The $\Delta$ notation is an older way to describe the derivative, but you'll still see it in places (usually without any accompanying explanation!) One reason math educators want to get rid of it is that students often think that $\Delta x$ means $\Delta$ times $x$. That is not the case; $\Delta x$ is one single quantity.

The closer point $B$ is to point $A$ the closer we are to seeing the rate of change of the function (the slope) exactly at point A. The exact slope is found by taking the limit of the process that brings the two points closer and closer together. Thanks to your introductory chapter on limits, you know how to use limits to find a function value at an awkward spot. Here we will use a limit to find the value of the slope. The two points we will be using are ( $x, x^{2}$ ), and a point just slightly to the right of that, $\left(x+\Delta x,\left(x+\Delta x^{2}\right)\right)$. To find the slope we take the difference between the $y$ coordinates and divide that by the difference in the $x$ coordinates. Then we imagine that these two points are closer and closer together. The result that we want is the limit as $\Delta x$ goes to zero ( $\lim _{\Delta x \rightarrow 0}$ ).

The rate of change $=$ the slope $=\lim _{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$. When the small changes $\Delta y$ and $\Delta x$ become infinitely small we call them $d y$ and $d x$, so you can just write the limit as the ratio $\frac{d y}{d x}$.
$\frac{d y}{d x}=\lim _{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}=\frac{(\mathrm{x}+\Delta \mathrm{x})^{2}-\mathrm{x}^{2}}{\mathrm{x}+\Delta \mathrm{x}-\mathrm{x}}=\lim _{\Delta \mathrm{x} \rightarrow 0} \frac{\mathrm{x}^{2}+2 \mathrm{x} \Delta \mathrm{x}+\Delta \mathrm{x}^{2}-\mathrm{x}^{2}}{\Delta \mathrm{x}}=\lim _{\Delta \mathrm{x} \rightarrow 0} \frac{2 \mathrm{x} \Delta \mathrm{x}+\Delta \mathrm{x}^{2}}{\Delta \mathrm{x}}=\lim _{\Delta \mathrm{x} \rightarrow 0} 2 \mathrm{x}+\Delta \mathrm{x}=2 \mathrm{x}$

Notice that a division is done first, and then we can take the limit.
When $\mathrm{x}=1$ the slope is 2 , and when $\mathrm{x}=2$ the slope is 4 .
As you learned when studying Limits, there is a left-hand limit and a right-hand limit. If you consider $\Delta x$ to be a positive quantity, then what you are really doing in the calculations above is finding the right-hand limit. $x+\Delta x$ is a little to the right of $x$. We will get the same limit, or derivative, if we start a little to the left of $x$, at $x-\Delta x$. Now our two points are ( $x, x^{2}$ ) and $\left(x-\Delta x,(x-\Delta x)^{2}\right)$. See if you can use these points along with some algebra to show that the derivative of $y=x^{2}$ is $2 x$.

## Function Notations for the Derivative

Notations for the derivative are:

$$
\frac{d y}{d x}, f^{\prime}(x), f^{\prime}, \text { or } y^{\prime} .
$$

You also need to be able to recognize the derivative when it is written as a limit:

$$
f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \text { or } f^{\prime}(a)=\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a}
$$

The derivative is the rate of change of a function. At first the derivative was just written as a ratio of differences (differentials). For $y=x^{2}, d y=2 x d x$, so $\frac{d y}{d x}=2 x$. Later on people began to use $f^{\prime}(x)$ to indicate the derivative of a function like $f(x)=x^{2}$, which makes sense because the derivative itself is also a function. If the original function is written as $y=x^{2}$, you can use $y^{\prime}$ to indicate its derivative. We say that $f^{\prime}(x)=2 x$, or $y^{\prime}=2 x$.

Again we will look at two points on the graph of $y=x^{2}$, but now we will consider this as the graph of the function $f(x)=x^{2}$, which looks more impressive than just calling it $y=x^{2}$. We will use the same method to find the exact rate of change of the function at some value of $x$, but now we'll use a different notation. The value of the function at $x$ is $f(x)$. To get the slope of the tangent line at ( $x, f(x)$ ), we will consider a point just slightly to the right of this, at ( $x+h, f(x+h)$ ). The slope of the line between these two points is $\frac{f(x+h)-f(x)}{x+h-x}$, which is the same as $\frac{f(x+h)-f(x)}{h}$.

The line between the two points is not a tangent line, but it will be when the two points are so close together that $\mathrm{h}=0$. But wait, h can't be 0 because we can't divide by zero. Well, just write it as limit:
$f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$
Now use the actual function, $f(x)=x^{2}$ :
$f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}=\lim _{h \rightarrow 0} \frac{(x+h)^{2}-x^{2}}{h}=\lim _{h \rightarrow 0} \frac{x^{2}+2 x h+h^{2}-x^{2}}{h}=\lim _{h \rightarrow 0} 2 x+h=2 x$

This is an improvement over the $\Delta$ notation used in the previous section, because it emphasizes that we are working with a function, and it actually looks a bit simpler. Note that $h$ could be a negative value since the limit may be approached from either the right or the left and it is the same either way.

Unfortunately, people couldn't just leave well enough alone. Here is an even more recent version of the same thing:

Suppose we want to find the derivative at some point on the graph of the function $f(x)=x^{2}$. The $x$-value of this point could be 1 , or 2 , or say " $a$ ". So the point on the graph we are interested in is ( $a, f(a)$ ), and we want the derivative there, which is $f^{\prime}(a)$. To get a second point to use to calculate a slope we will take a generic point $(x, f(x))$ on the graph a little to the right of $(a, f(a))$. We would calculate the slope of the line between these points by writing $\frac{f(x)-f(a)}{x-a}$, in that order because we usually put the right point first. The order is actually arbitrary and point $(x, f(x))$ could be to the left of $(a, f(a))$. The limit can be approached from the right or the left, and it is the same either way.

Just like we did before, we want to move these points infinitely close together so we can get the exact slope at a single point. That means we want $x-a$ to get smaller and smaller until the two points touch each other. Once they do, $x=a$. We look for the limit as $x$ approaches $a$.

First, we write that the slope at point $(\mathrm{a}, \mathrm{f}(\mathrm{a}))$ is
$\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a}$
For this function, $f(x)=x^{2}$, that means that the slope at $(a, f(a))$ is
$f^{\prime}(a)=\lim _{x \rightarrow a} \frac{x^{2}-a^{2}}{x-a}$
Now we can use a different trick to find the derivative: the difference of two squares. We get
$f^{\prime}(a)=\lim _{x \rightarrow a} \frac{(x+a)(x-a)}{x-a}=\lim _{x \rightarrow a} x+a=a+a=2 a$
At any point $\left(a, a^{2}\right)$ the rate of change of the function is $2 a$, so for any point ( $x, x^{2}$ ) the rate of change is $2 x$. This is the same conclusion we reached twice before. However, this is obviously better because we got to use the difference of two squares

You may encounter still more different ways to write the derivative. I would highly recommend
that you pick your own "favorite" way so that you can feel comfortable when a problem is presented that way, and superior when it is not, because your way is better.

You should learn to write the derivative as $\frac{d y}{d x^{\prime}} f^{\prime}(x), f^{\prime}$, or $y^{\prime}$. You also need to be able to recognize the derivative when it is written as a limit:
$f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$
$f^{\prime}(a)=\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a}$
On tests, derivatives may look like $\lim _{h \rightarrow 0} \frac{\sin \left(\frac{\pi}{2}+h\right)-\sin \left(\frac{\pi}{2}\right)}{h}$, or even like this:
$\lim _{x \rightarrow 2} \frac{\arctan (2 x)-\arctan (4)}{x-2}$.
It is very important that you recognize these expressions as derivatives, and do not try to solve them as you did in the chapter on limits. Even if you can solve them as a limit, you'll waste valuable test time.

## With Respect to x ...

We usually consider the change in the value of a function relative to the change in $x$ : the derivative is $\frac{d y}{d x}$. If $y$ is a function of the time $t$, then the derivative is $\frac{d y}{d t}$.

Your calculus instructor (or textbook) may use the phrase "with respect to $x$ " a lot, so it is important that you really understand what it means. No actual respect for the variable x is
involved here. For any given function of $x$, the output changes depending on the change in $x$. Look at the graph below, and try to describe how y is changing as you move along the line.


Your description should contain something like " $y$ increases by 3 units as $x$ increases by one unit". Because this is a constant change you don't actually need calculus, but calculus does work just as well here. This is the graph of $y=3 x$. We say that the derivative of $y$, with respect to $x$, is 3 . By this we mean 3 units of increase per unit increase in $x$. This constant rate of change doesn't depend on the value of $x$. It is the same everywhere along the graph. We write the derivative as $\frac{d y}{d x}=3$, which shows that we mean that the change in $y$ divided by the change in x is 3 . This notation emphasizes that the derivative is a ratio. The derivative is the slope of the tangent line, but here the tangent line is just the function itself, and the slope is 3 .

Notice that this will work for any linear function. If $y=a x$, then $\frac{d y}{d x}=a$.

Now let's take the derivative with respect to x of the constant function $\mathrm{y}=10$. How much does $y$ change as $x$ increases by 1 unit? The answer is that it doesn't change at all. The derivative $\frac{d y}{d x}$ is 0 . If $y=c$, then $\frac{d y}{d x}=0$.

## How Does a Circle Grow?

For $A=\pi r^{2}, \frac{d A}{d r}=2 \pi r$. The rate of change of the area is equal to the circumference.

Now let's look at a real-life derivative.
The area of a circle is given by $A=\pi r^{2}$.
As the radius increases, so does the area. We want to know how much the area increases per unit increase in the radius. As it turns out, when the radius is small the area grows more slowly than when the radius is larger. Let's take a circle with a radius of length $r$. The area of this circle is $\pi r^{2}$. Now increase the radius just a little bit to $r+h$. Then the area is $\pi(r+h)^{2}$. The rate of increase of the area per unit increase in $r$ is given by the change in the area divided by the change in the radius. We can think of a circle growing by adding an extremely thin ring around its circumference. The area of this thin ring is the area of the new circle minus the area of the old circle: $\pi(r+h)^{2}-\pi r^{2}$. We divide by the change in the radius: $r+h-r$.
$\frac{\pi(\mathrm{r}+\mathrm{h})^{2}-\pi r^{2}}{\mathrm{r}+\mathrm{h}-\mathrm{r}}=\frac{\pi \mathrm{r}^{2}+2 \pi \mathrm{rh}+\pi \mathrm{h}^{2}-\pi \mathrm{r}^{2}}{\mathrm{~h}}=\frac{2 \pi \mathrm{rh}+\pi \mathrm{h}^{2}}{\mathrm{~h}}=2 \pi \mathrm{r}+\pi \mathrm{h}$.


The limit of this as h goes to zero is $2 \pi r$. The derivative of the area with respect to the radius, $\frac{\mathrm{dA}}{\mathrm{dr}}$, is $2 \pi r$, which is the circumference. If you think of the radius as constantly increasing, you can see how the area of the circle increases. When $r$ is one unit, like maybe 1 inch, the area grows at a rate of $2 \pi \cdot 1$, which is about 6.28 . The area is increasing by 6.28 square inches per inch of radius. By the time the radius is 2 inches, the area of the circle is growing by $2 \pi \cdot 2$, or about 12.56 square inches per unit of radius. The area increases faster and faster.

## When Not to Use Derivatives

The function must be continuous at the given point: Check for an asymptote, and make sure that $f(x)$ has the same value whether you approach from the left or from the right.

The function must be differentiable at the given point: Check that the derivative has the same value whether you approach from the left or from the right.

A function may have a vertical tangent line (no derivative) at one or more points.

## 1. Discontinuity

By definition, a function that is not continuous at a point has no derivative there. That is rather obvious for a jump discontinuity, which creates a big gap at that point. If you have a piece-wise defined function but no graph is supplied, you should first check if both parts have the same function value at the point you want to take the derivative at. That will probably mean inserting an $x$-value just outside the stated limit for one of the pieces Please note that if your function does turn out to be continuous at the given point, it will not necessarily have a derivative there (see next section).

Sometimes the Limit of the function value at a particular point does not exist because the function value keeps getting larger and larger or smaller and smaller near this point. That is an infinite discontinuity. If you tried to draw a tangent line at such a point, you would see that it would be vertical.

If a function has a removable discontinuity at a point, then by definition it has no derivative at that point, even though we can use limits to find the function value. The derivative is the limit as $h$ goes to zero of $\frac{f(x+h)-f(x)}{h}$, so if $f(x)$ doesn't exist that is a nonstarter. Also, you would run afoul of the definition of "tangent line" since the line wouldn't be touching the curve at a point. A function like $f(x)=\frac{(x-2)(x+5)^{2}}{x-2}$ has no derivative at $x=2$. That value for $x$ is not in the domain of the function. Remove the discontinuity first so you can just find the derivative in the usual way.

## 2. Sudden Change in Direction

Some functions contain sharp corners or cusps. An example would be $f(x)=|x|$. For a linear function, the slope of the tangent line is just the function line itself. If you approach $x=0$ from the right, the slope is 1 , while if you approach $x=0$ from the left the slope is -1 . This tells you that $\mathrm{x}=0$ is not a good spot to try to find the derivative. Check that the derivative is the same whether you approach the given point from the left or from the right.

## 3. Vertical Tangent

From your study of limits, you would have seen that it is not always possible to find the slope of a tangent line. If a continuous function has a vertical tangent line at some point then there is no slope. Vertical lines have no slope (slope $=$ rise/run, and the run is 0 ). The function has no derivative at this particular point.

## IV. Working with Derivatives

## Basic rules for derivatives:

1. The derivative of a single constant is 0 . If $y=5$, then $\frac{d y}{d x}=0$
2. The derivative of a linear function is a constant. If $y=4 x$, then $\frac{d y}{d x}=4$.
3. If the function has parts that are separated by a + or - sign, take the derivative of each part separately and add or subtract the individual derivatives.

## Example

Find the derivative of the function $y=7 x-10$.
Derivatives can just be added or subtracted. The derivative of 7 x is 7 , and the derivative of 10 is 0 . Subtract the two derivatives: $\frac{\mathrm{dy}}{\mathrm{dx}}=7-0=7$.

## The Power Rule

$$
\text { For } y=x^{n}, \frac{d y}{d x}=n x^{n-1}
$$

The Power Rule says that if $y=x^{n}$, then $\frac{d y}{d x}=n x^{n-1}$. So, if $y=x^{3}$, then $\frac{d y}{d x}=3 x^{2}$. If there is a constant in front of $x$, it stays there: for $y=5 x^{3}, \frac{d y}{d x}=15 x^{2}$.

The Power Rule works for negative and fractional exponents too. To take advantage of this, replace any radicals with fractional exponents.

## Example

If $y=\sqrt{x}$, find $\frac{d y}{d x}$.
Rewrite $\sqrt{\mathrm{x}}$ as $\mathrm{y}=\mathrm{x}^{\frac{1}{2}}$. Now apply the Power Rule: $\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{1}{2} \mathrm{x}^{-\frac{1}{2}}$.
Remember that a negative exponent means 1 divided by the variable to that power. $\mathrm{x}^{-\frac{1}{2}}$ means $\frac{1}{x^{\frac{1}{2}}}$ which is $\frac{1}{\sqrt{x}}$. And $\frac{1}{2} \cdot \frac{1}{\sqrt{x}}=\frac{1}{2 \sqrt{x}}$.

Because of the Power Rule, you'll be working with exponents a lot, and especially with fractional exponents. Although fractional exponents may look intimidating, they still have to follow the same rules as regular exponents. Those rules are shown in the box below. If that doesn't look familiar, please check out "Reference: Exponents" in the first chapter.

$$
\begin{array}{ll}
\hline x^{3} \cdot x^{4}=(x \cdot x \cdot x) \cdot(x \cdot x \cdot x \cdot x)=x^{7} . & \text { General rule: } \\
\frac{x^{a}}{x^{3}}=\frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x}=x^{2} . & \frac{x^{a}}{x^{b}}=x^{a+b} \\
\frac{x^{3}}{x^{5}}=x^{3-5}=x^{-2}=\frac{x \cdot x \cdot x \cdot 1}{x \cdot x \cdot x \cdot x \cdot x}=\frac{1}{x \cdot x}=\frac{1}{x^{2}} \cdot & x^{-a}=\frac{1}{x^{a}}(x \neq 0) \\
\frac{x^{3}}{x^{3}}=x^{3-3}=x^{0}=1 . & x^{0}=1 \quad(x \neq 0) \\
\left(x^{4}\right)^{3}=x^{4} \cdot x^{4} \cdot x^{4} . & \left(x^{a}\right)^{b}=x^{a b} \\
(x y)^{3}=x y \cdot x y \cdot x y=x^{3} y^{3} \cdot & (x y)^{a}=x^{a} y^{a} \\
x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} \cdot x^{\frac{1}{3}}=x^{1}, \text { so } x^{\frac{1}{3}}=\sqrt[3]{x} & x^{\frac{1}{n}}=\sqrt[n]{x}
\end{array}
$$

$$
x^{\frac{2}{3}}=\left(x^{2}\right)^{\frac{1}{3}} \text { or }\left(x^{\frac{1}{3}}\right)^{2} \quad x^{\frac{a}{n}}=\sqrt[n]{x^{a}} \text { or }(\sqrt[n]{x})^{a}
$$

## The Chain Rule

Take the derivative of the outer function, and multiply it by the derivative of the inner function.

If $h(x)=f(g(x))$, then $h^{\prime}(x)=f^{\prime}(g(x)) \cdot g^{\prime}(x)$.

Most students find the Chain Rule rather intuitive. To see how it works, Let's find the derivative of something that we could easily manage without the Chain Rule:

## Example

Find the derivative of $y=(3 x)^{2}$
Because we can square this first, to get $9 x^{2}$, we already know that the derivative should be $18 x^{2}$.
To apply the chain rule, think of $(3 x)^{2}$ as an "outer" function, $y=(u)^{2}$, and an "inner" function, $u=3 x$.

First, take the derivative of $u^{2}$ :
$\frac{d y}{d u}=2 u$
Since $u$ is really $3 x, \frac{d y}{d u}=2(3 x)$
Next, take the derivative of the inner function:
$\frac{d u}{d x}=3$.

Now we have $\frac{\mathrm{dy}}{\mathrm{du}}$ and $\frac{\mathrm{du}}{\mathrm{dx}}$, but what we really want is $\frac{\mathrm{dy}}{\mathrm{dx}}$. Just multiply the two derivatives:
$\frac{d y}{d x}=\frac{d y}{d u} \cdot \frac{d u}{d x}$
$\frac{d y}{d x}=2(3 x) \cdot 3$
$\frac{d y}{d x}=18 x$
When you multiply the two derivatives like this, du cancels out. This is the Chain Rule, and it is quite easy to use. Here you can see how the derivative is a real ratio that expresses the change in one quantity relative to the change in another quantity.

So, for $y=(3 x)^{2}$ the derivative is $2(3 x) \cdot 3$, which is $18 x$.

## Example

Find the derivative of $y=\left(x^{2}+5\right)^{3}$.
Here you wouldn't want to go to the trouble of actually raising the whole expression to the third power, although you could do that. Let's think of this as two functions:

The "outer" function is $y=(\ldots)^{3}$, and the "inner" function is $x^{2}+5$
The derivative of the outer function is $3\left(x^{2}+5\right)^{2}$, and the derivative of the inner function is 2 x :
$\frac{d y}{d x}=3\left(x^{2}+5 x\right)^{2} \cdot(2 x)=6 x\left(x^{2}+5 x\right)^{2}$

## Example

Find the derivative of $h(x)=f(4 x)$
Things often look more confusing when they are phrased in an abstract way. Of course we are still free to change this abstract problem to something more specific. Instead of $h(x)$ l'll write $y$, just because it looks simpler. Now l'll make up some simple random function to represent $f(x)$, like $f(x)=x^{2}$. This is the "outer" function. Inside the parentheses we see another function: $4 x$. This doesn't need to have a name, although you could call it $\mathrm{g}(\mathrm{x})$. Now I can write my sample function as $y=(4 x)^{2}$. To take the derivative of that, I would write $y^{\prime}=2(4 x) \cdot 4$.

For $h(x)=f(4 x)$, we can write the derivative like this:
$h^{\prime}(x)=f^{\prime}(4 x) \cdot 4$
$h^{\prime}(x)=4 f^{\prime}(4 x)$

## The Product Rule

The derivative of the product of two functions is the derivative of the first, multiplied by the second, plus the first multiplied by the derivative of the second.

If $y=f(x) \cdot g(x)$, then $y^{\prime}=f^{\prime}(x) \cdot g(x)+f(x) \cdot g^{\prime}(x)$

The Product Rule may seem a bit mysterious. Why can't we just take the derivatives of each part and multiply them? Actually, the original idea of differences (differentials) shows quite clearly why and how the product rule works.

A line with length x grows by adding infinitely tiny amounts dx to the length:

A square grows by adding infinitely tiny amounts dx to both sides:



The infinitely tiny change in the area is dA. The square grows by adding two infinitely thin strips to two of its sides. The increase in area is the area of those strips. The width of each strip is dx . The length of one side of the strip is $x$, and on the other side it is $x+d x$ because one of the ends of the strip is slanted. If we claim that $x+d x=x$, we can simply use $x$ as the length. The area of each strip is $x d x$, which means that $d A=2 x d x$.

To get the derivative, just divide both sides by dx :
$\frac{\mathrm{dA}}{\mathrm{dx}}=2 \mathrm{x}$
So, if $y=x^{2}$, then $\frac{d y}{d x}=2 x$

Now let's see how a rectangle grows when both of the sides are changing. That is slightly more complex because the length is different from the width. We'll call the area of the rectangle $y$. The area is the product of the sides, which we will call $u$ and $v . y=u v$ :


Let $u$ increase by an infinitely small amount $d u$, and $v$ by an infinitely small amount $d v$ (du and dv may be different in size):


This only shows the actual change in y , which is dy . dy is the sum of the red strip and the blue strip: $d y=u d v+v d u$.

To get a derivative, which is a rate of change, we need to decide what to measure that change by, that is, we need the rate of change of $y$ with respect to something. Both $u$ and $v$ could change with time, so you can figure out how the area $y$ is changing over time. More commonly,
$u$ and $v$ will both be functions of $x$. They both change as $x$ changes, and so does the area. The derivative will be the change relative to the change in $x$. An infinitely tiny change in $x$ is called $d x$. If $d y=u d v+v d u$, then we can get the derivative $d y / d x$ by dividing both sides of the basic product rule equation by dx :
$\frac{d y}{d x}=u \frac{d v}{d x}+v \frac{d u}{d x}$
As you will see when you study the quotient rule, the product rule is best turned around:
$\frac{d y}{d x}=\frac{d u}{d x} v+u \frac{d v}{d x}$
Read it as: "The derivative of the first function times the second, plus the first function times the derivative of the second."

In shorthand notation it looks a little simpler: $y^{\prime}=u^{\prime} v+v^{\prime} u$
Because $u$ and $v$ are functions of $x$, the product rule is often written like this:
If $y=f(x) \cdot g(x)$, then $y^{\prime}=f^{\prime}(x) \cdot g(x)+f(x) \cdot g^{\prime}(x)$

Or use the shorter form: $y^{\prime}=f^{\prime} g+f g^{\prime}$

## Example

$y=5 x\left(x^{2}+2\right)$, what is the derivative, $\frac{d y}{d x}$ ?
Here the first function is $5 x$, and the derivative of that is 5 .
The second function is $x^{2}+2$, so multiply the derivative of the first function by the second function to get $5\left(x^{2}+2\right)$

Now add the derivative of the second function multiplied by the first. The derivative of $x^{2}+2$ is $2 x$, which should be multiplied by $5 x$.
$\frac{d y}{d x}=5\left(x^{2}+2\right)+5 x \cdot 2 x$
$\frac{d y}{d x}=5 x^{2}+10+10 x^{2}$
$\frac{d y}{d x}=15 x^{2}+10$
If we are correct, this derivative should be the same as the one we get by multiplying first and then taking the derivative:
$y=5 x\left(x^{2}+2\right)=5 x^{3}+10 x$
$\frac{d y}{d x}=15 x^{2}+10$, and $y e s$, it is the same.

A product may involve a constant, as in $y=3 x$. Do we have to use the Product Rule here? No, we don't need to. Remember that $\mathrm{y}=3 \mathrm{x}$ is a straight line, and the derivative (the change in y relative to the change in $x$ ) is just the slope, 3 . You can also look at that in a different way:

The area of the rectangle below is $y$, which is $3 x$. If $x$ grows by an infinitely small amount $d x$, the area of the rectangle will grow by an infinitely thin strip along one side. That strip is shown as a blue line in the picture below:


The area of the thin blue strip is the width times the length, or $d x$ times 3 . Notice that it is only $x$ that changes, not 3 . $d y=3 d x$. Divide both sides by $d x$ to $g e t \frac{d y}{d x}=3$.

The Product Rule may not be necessary here, but it still works. Let's try to find the difference of $3 x$ by using the Product Rule.

If $y=u v$, then $y^{\prime}=u^{\prime} v+v^{\prime} u$
Here $u$ will be 3 , and $v$ will be $x$. Because 3 is a constant, $\frac{d u}{d x}$ will be zero. $\frac{d v}{d x}$ is 1 , because the derivative of $x$ is 1 . That last part should make sense to you because the change in $x$ relative to the change in $x$ has to be 1 .
$y=3 x$
$y^{\prime}=0 \cdot x+1 \cdot 3$
$y^{\prime}=3$
Normally we only use the Product Rule for situations where both $u$ and $v$ are changing, as in the previous example. This also tells us that for $y=5 x\left(x^{2}+2\right)$, we could just take the derivative of $x\left(x^{2}+2\right)$ and multiply the result by 5 .

## The Quotient Rule

$$
\text { If } y=\frac{f(x)}{g(x)} \text { then } y^{\prime}=\frac{f^{\prime}(x) \cdot g(x)-f(x) \cdot g^{\prime}(x)}{g(x)^{2}}
$$

Let's see what happens to the change in $y$ when there is a quotient instead of a product, like maybe $y=\frac{5 x}{x^{2}+2}$.

It is not difficult to rewrite this so that you can use the Product Rule and the Chain Rule: $y=5 x\left(x^{2}+2\right)^{-2}$. However, there is also a Quotient Rule that you can use to find the derivative when there is some function $u$ divided by some other function $v$.

For $y=\frac{u}{v}$, where $u$ and $v$ are functions of $x$, you can write $y=u \cdot \frac{1}{v}$. The infinitely tiny change in $y$, $d y$, can be expressed in terms of the infinitely tiny change in $u$ and the infinitely tiny change in $\frac{1}{v}$ by using the product rule. Since $\frac{1}{v}$ can be written as $v^{-1}$, its difference is $-v^{-2} d v$, or $-\frac{1}{v^{2}} d v$. As $v$
increases by an infinitely tiny bit $\mathrm{dv}, \frac{1}{\mathrm{v}}$ decreases by $\frac{\mathrm{dv}}{\mathrm{v}^{2}}$. u just increases by du , as shown in the picture below:

$$
-\frac{d v}{v^{2}}
$$



As both $u$ and $v$ increase, the total change in the rectangle is $\frac{1}{v} \cdot d u+u \cdot-\frac{d v}{v^{2}}$, which is the same as $\frac{\mathrm{du}}{\mathrm{v}}-\frac{\mathrm{udv}}{\mathrm{v}^{2}}$. That's a bit ugly, but we can put it over a common denominator. Multiply $\frac{\mathrm{du}}{\mathrm{v}}$ by $\frac{\mathrm{v}}{\mathrm{v}}$ to change it: $\frac{\mathrm{v}}{\mathrm{v}} \cdot \frac{\mathrm{du}}{\mathrm{v}}=\frac{\mathrm{vdu}}{\mathrm{v}^{2}}$.
$\frac{\mathrm{vdu}}{\mathrm{v}^{2}}-\frac{\mathrm{udv}}{\mathrm{v}^{2}}$

Put that all together to find that $d y$, the difference of $\frac{u}{v}$, is $\frac{v d u-u d v}{v^{2}}$. This is the original Quotient Rule.

To get the derivative, divide both sides by dx . That is the same as multiplying everything by $\frac{1}{\mathrm{dx}}$ :
$\frac{d y}{d x}=\frac{v d u-u d v}{d x v^{2}}$
That's good, but we want to be able to see this in terms of the derivative of the functions $u$ and $v$. That is, we want to see $\frac{d u}{d x}$ and $\frac{d v}{d x}$ in our formula. To make that happen, divide both the top and bottom of the fraction by dx , which makes dx cancel out on the bottom:
$\frac{d y}{d x}=\frac{v \frac{d u}{d x}-u \frac{d v}{d x}}{v^{2}}$
Using the simpler ' notation for the derivative ( $u^{\prime}$ instead of $\frac{\mathrm{du}}{\mathrm{dx}}$ ) the quotient rule looks like this:
$\mathbf{y}^{\prime}=\frac{\mathbf{u}^{\prime} \mathbf{v}-\mathbf{u} \mathbf{v}^{\prime}}{\mathbf{v}^{2}}$
Or, in function notation: If $y=\frac{f(x)}{g(x)}$ then $y^{\prime}=\frac{f^{\prime}(x) \cdot g(x)-f(x) \cdot g^{\prime}(x)}{g(x)^{2}}$
Take the derivative of the top function multiplied by the bottom function, subtract the top function multiplied by the derivative of the bottom function, and divide the whole thing by the square of the bottom function.

Using the Quotient Rule is not actually difficult, but it tends to get messy so you can make mistakes. Simplifying the resulting derivative can be quite time-consuming and error-prone.

Survival Tip: Practice with some easy problems first before trying the harder ones.

## Example

$y=\frac{5 x}{x^{2}+2}$. Find the derivative by using the Quotient Rule.
As we said, we can call the top function $u$ and the bottom function $v$. The derivative of the top function, $5 x$, is just 5 . The derivative of the bottom function, $x^{2}+2$, is $2 x$. Now fill in the formula:
$y^{\prime}=\frac{u^{\prime} v-u v^{\prime}}{v^{2}}$
$y^{\prime}=\frac{5\left(x^{2}+2\right)-5 x(2 x)}{\left(x^{2}+2\right)^{2}}$
$y^{\prime}=\frac{5 x^{2}+10-10 x^{2}}{\left(x^{2}+2\right)^{2}}=\frac{-5 x^{2}+10}{\left(x^{2}+2\right)^{2}}$

## Finding a Tangent Line

$y-y_{1}=m\left(x-x_{1}\right)$, where $\left(x_{1}, y_{1}\right)$ is the given point, and $m$ is the derivative at $x_{1}$.
$y_{1}$ may not be supplied, but can be found from the given function by using $x_{1}$.

A problem may ask you to find a tangent line to a curve at a particular point. Let's reword that as "create a tangent line to a curve", because it sounds more constructive and puts you in control.

Creating a tangent line is not hard at all, but once your paper is cluttered up with derivatives it is easy lose track of what you are doing. Make the tangent line first, before you do any calculus.

From your study of algebra and geometry, you already know how to create the equation of a line that passes through a given point. All you need is one point and the slope. As you may remember, straight lines can always be described by the equation $y=m x+b$. This equation will hold for any point on the line, so it will also work for the point given in the problem. Let's call that point ( $\mathrm{x}_{1}, \mathrm{y}_{1}$ ). Now we have:
$y=m x+b$
$y_{1}=m x_{1}+b$
Subtract the second equation from the first, to eliminate the unknown b:
$y-y_{1}=m x-m x_{1}$

Factor out $m$ to get the point-slope form of the equation:
$y-y_{1}=m\left(x-x_{1}\right)$
This is called the point-slope form because all you need to do is plug in a point and a slope.
First, find the point and plug it in. Quite possibly your problem will only give you the $x$ coordinate of the point. If you haven't covered your paper with a pile of calculus that contains multiple $x$ 's and $y$ 's, the $y$-coordinate will be easy to find. Just plug $x$ into the equation of the function and get $y$. Then take your point and put it into the point-slope form of the line. For example, if your point is $(5,7)$, then the line will look like this:
$y-7=m(x-5)$
There, that just about does it. All you need now is the slope $m$. Here is where you start using calculus. The derivative gives you the slope of the tangent line, for any value of $x$. Once you have the derivative, make sure to use the right value of $x$, which is 5 in this case, to get a numerical value for the slope. That number is $m$. Plug it into the point-slope form. Other than rearranging it a bit to make it look pretty, you're done. When you are finished, use a graphing program or calculator to draw both the original function and the tangent line. You may not be able to see the exact point where they touch, but it should look about right.

## Example

Create the tangent line to the curve $\mathrm{y}=\mathrm{x}^{2}+2$ at $\mathrm{x}=4$.
When $x=4, y=4^{2}+2=18$. The point is $(4,18)$.
Put the point into the point-slope form equation for the line:
$y-y_{1}=m\left(x-x_{1}\right)$
$y-18=m(x-4)$
Now start calculus. Find the derivative of $y=x^{2}+2$, which is $2 x+0$, or simply $2 x$. That tells you that the slope of the tangent line is $2 x$, for any value of $x$. At the point where we want the tangent line to appear $x$ is 4 , so the slope of the tangent will be 8 . Stick that into your equation:
$y-18=m(x-4)$
$y-18=8(x-4)$
That's it, but you can make it look nicer:
$y-18=8 x-32$
$y=8 x-14$

## Example

A line that is tangent to the curve $y=x^{2}-4 x+1$ has a slope of 2 . Find the point of tangency.

The point of tangency is where the line just touches the curve. The slope of the line is determined by the derivative of the function at that point. First find the general derivative:
$\frac{d y}{d x}=2 x-4$
At some point, the value of the derivative (the slope of the line) will be 2 :
$2 x-4=2$

Solve that to get $x=3$. To find the $y$ value, look at the original curve $x^{2}-4 x+1$. When $x$ is $3, y$ is -2 , so the point of tangency is $(3,-2)$.

## Linearization and Linear Approximation

First, create a tangent line at the closest known point: $y-y_{1}=m\left(x-x_{1}\right)$.
Use the derivative at that point to find the slope $m$.

Use your tangent line to find the approximate y value you need.

As we said earlier, mathematicians are happy to be able to find tangent lines. One of the things they can do with these tangent lines is to find the approximate value of a function at a point where it is otherwise difficult to find such a value. Although we now have calculators and
computers that can do this faster, it is still useful in some situations. What we will do to estimate the function value at a particular point is to draw a tangent line at a convenient nearby point. This tangent line will be very close to the function curve for the point we need. Instead of finding the function value at that point, which is hard, we simply find the $y$-value of the tangent line at that point, which is easy. Look at the example below to see how it works:

## Example

Find $\sqrt{3.9}$ using linear approximation. Compare your result with the value obtained by using a calculator.

The closest $x$ value for which it is easy to calculate a square root is $x=4$. We draw a tangent line to the curve $y=\sqrt{x}$ at the point $(4,2)$, marked $A$ in the picture below. The tangent line doesn't diverge far from the curve when $x=3.9$, so we draw point $B$ on the tangent line here, very close to the point
$(3.9, \sqrt{3.9})$ on the curve. The square root of 3.9 is hard to calculate, but the $y$ value of point $B$ is very easy to find because the tangent line will have a simple linear equation. The equation of the tangent line is called the linearization of the function $f(x)=\sqrt{x}$. The linearization is different at every point x along the curve.

below is a close-up view of the same situation, showing that point $B$ is located on the tangent line, while point $A$ is located both on the curve and on the tangent line (the tangent line touches the curve at point $A$ ). The green line represents the size of the error we will make by using the $y$-value of point $B$ to estimate the $y$ value of the function. The error will be larger when $B$ is further away from $A$. Also notice that we are overestimating the value of $\sqrt{3.9}$ here because the sample curve is concave down. If we use this same procedure with a curve that is concave up our estimate would be below the actual value.


Now all you need to do is construct the tangent line. Make the line before doing any calculus. Start with the general equation $y-y_{1}=m\left(x-x_{1}\right)$, where $m$ is the slope. Plug in the point of tangency, which is point $A,(4,2) . y-2=m(x-4)$. There, we have a line. All we need now is the slope, and for that we use calculus.

Remember that the slope of the tangent line will be the derivative of the function at that point. To find the derivative of $y=\sqrt{x}$, rewrite it as $y=x^{1 / 2}$. We see that $\frac{d y}{d x}=\frac{1}{2} x^{-1 / 2}$. Since $x^{-1 / 2}=\frac{1}{x^{\frac{1}{2}}}=$ $\frac{1}{\sqrt{x}}$, we write $\frac{1}{2} x^{-1 / 2}$ as $\frac{1}{2 \sqrt{x}}$.
$\frac{d y}{d x}=\frac{1}{2 \sqrt{x}}$, so when $x=4$ we get $\frac{d y}{d x}=\frac{1}{4}$. The slope of the tangent line is $\frac{1}{4}$.

Substitute that into the formula:
$y-2=\frac{1}{4}(x-4)$. This is the equation of the tangent line, and since the $x$ we want is very close to 4 we won't bother to rewrite this equation in the form $\mathrm{y}=\mathrm{mx}+\mathrm{b}$. Instead we can use it just as it is, since it gives the relationship between $y$ and $x$ for any point on the tangent line. To get point $B$ we need $y$ when $x=3.9$, so put that into the equation:
$y-2=\frac{1}{4}(3.9-4)$
$y-2=\frac{1}{4}(-.1)$
$y-2=-.025$
$y=2-.025=1.975$.
Now we have the coordinates of point $B,(3.9,1.975)$ on the tangent line, and point $B$ is really close to the point $(3.9, \sqrt{3.9})$ on the function curve.

It should come as no surprise that the $y$ value of point $B, 1.975$, is very close to the answer you get from a calculator for $\sqrt{3.9}$. It is just slightly larger than the actual value because the tangent line is just a little bit above the curve.

You can see that since we expect $x$ and $x_{1}$ to be close together in this type of problem, the form $y-y_{1}=m\left(x-x_{1}\right)$ has the advantage of making our calculation easier. However, you can do this any way you prefer, so if you like to use $y=m x+b$ that works too.

## Derivative Graphs

The derivative of a function is also a function. The derivative function describes how the original function is changing.

Make sure you can relate the graph of the derivative to the graph of the original function.

We have seen that the derivative of the function $y=x^{2}$ is $2 x$, which means that the actual value of the derivative depends on the size of $x$. The derivative is different at every point on the curve described by $y=x^{2}$, and at every point the derivative describes how fast the curve is changing as $x$ changes. The image below shows the graph of the parabola $y=x^{2}$. You can see its derivative as the slope of the tangent lines drawn to the curve at different points:


At point $A$, the $y$-value of the function is decreasing as $x$ increases, and it is actually decreasing quite rapidly. This means that the derivative is negative, and that it has a large negative value. You can see that the slope of the tangent line at point $A$ is negative, and that the slope is quite steep. The $x$-value at point $A$ is -2 , so the derivative (the slope of the tangent line) is -4 . By the time we reach point $B$, the function is still decreasing, but not as rapidly. The derivative is still negative, but the slope of the tangent line has a smaller negative value, which is -2 . Once we pass point $C$, the function begins to increase so its derivative is positive. As x gets larger, the function increases faster and faster, so the derivative is also getting larger. Notice that the derivative is zero at point C because the function stops decreasing and starts increasing. This is the point where the derivative actually changes from being negative to being positive. The tangent line is horizontal (it has a slope of zero). At point E the slope of the tangent line is 4 .

The derivative is also a function, and we can graph it. This derivative function is $y=2 x$, which is just a straight line:


Wherever the derivative is negative, the original function is decreasing, and where the derivative is positive the original function is increasing. At point $A$, the original function is decreasing faster than at point $B$. At point $C$, the original function reaches its minimum value. You may want to spend a bit of time studying these last two graphs to make sure you understand the relationship between them, because that can be rather confusing at first.

## Derivatives of Trigonometric Functions

|  |  |
| :--- | :--- |
| Trig Function | Derivative |
| $\sin x$ | $\cos x$ |
| $\cos x$ | $-\sin x$ |
| $\tan x$ | $\sec ^{2} x$ |
| $\cot x$ | $-\csc ^{2} x$ |

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sec}x\quad\operatorname{sec}x\operatorname{tan}
CSC X -CSC X Cot X
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Since the derivative is really the rate of change of a function, we can look at a function graph and estimate the value of the derivative at any given point. Consider the sine function, $f(x)=$ $\sin (x)$. The amazing thing is that there is in fact a function that perfectly describes the derivative of the sine function at every point. It gives the exact value of the slope of the tangent line for every value of $x$. This function is $f(x)=\cos (x)$. Graph this function on the same screen as the sine function. Study both functions carefully until you can see that the cosine function is really the derivative of the sine function.


At point $A$, the sine function has reached its maximum value of 1 . The tangent line here would be horizontal, so the slope would be zero. The cosine function does in fact have a value of 0 at the same value of $x$. At point $B$, the sine function is decreasing at its most rapid rate. The tangent line would have a slope of -1 , which is also the value of the cosine function here. When we reach point C , the sine function is increasing, but it is doing so quite slowly. Correspondingly, the cosine function has a small positive value.

Now we should take some time out to consider a very important point. The sine and cosine functions that you are looking at are graphed in radians. If you try to graph them in degrees, the picture changes. You may think that you could just replace $\pi$ radians by 180 degrees, but then what would you do with the scale on the $y$-axis? From the unit circle, we know that a value of 1 for the sine or cosine corresponds to 1 radius of the unit circle. There are $2 \pi$ such radii in the circumference of the circle, and this is where we get the units we call radians. If you make the scale on the $y$-axis correspond to a degree scale on the $x$-axis, the unit 1 suddenly
becomes very much smaller because 1 degree is very tiny. The sine function really flattens out so that you can barely see it going up or down. So does the cosine function. Try it out by setting your calculator to degrees (use the MODE button) and graphing $y=\sin x$ and $y=\cos x$. The functions are now so flat that you can't even see them properly. This also changes the slope relationship so that the value of the cosine function no longer corresponds to the slope of the sine function. Once we start using degrees, the derivative of the sine is not equal to the cosine. To experiment with this, you can graph the sine and cosine functions in degrees by using $y=\sin \left(\mathrm{pi}^{*} \mathrm{x} / 180\right)$ and $\mathrm{y}=\cos \left(\mathrm{pi}^{*} \mathrm{x} / 180\right)$. This makes the proper adjustment to the x values. Note that the slope of the line tangent to the sine function is now the derivative of $\sin$ ( $\pi x / 180$ ), which is $\frac{\pi}{180} \cos (\pi x / 180)$.

If you plan to use your calculator for calculus, set it to radian mode right now!

The fact that the cosine function is the derivative of the sine function also illustrates an important point: the derivative is a function in itself. As a result, we can take the derivative of a derivative, which is called the second derivative. The second derivative is often written as $\mathrm{f}^{\prime \prime}(\mathrm{x})$. If you study the sine and cosine graphs long enough, you might notice that we can draw a graph that represents the derivative of the cosine function. This is actually the graph of $f(x)=-\sin (x)$. Taking the derivative of this, we get $f(x)=-\cos (x)$. The derivative of $f(x)=-\cos (x)$ is $f(x)=\sin (x)$, so then we are back where we started.


Once you know that the derivative of $\sin (x)=\cos (x)$, you will be expected to find the derivative of more complicated functions like $y=\sin \left(x^{2}\right)$. This looks like a good place to apply the chain rule, since there is an outer function $\sin (u)$, and an inner function $u=x^{2}$ :
$y=\sin (u)$

$$
\begin{aligned}
& \frac{d y}{d u}=\cos (u) \\
& u=x^{2}, \operatorname{so} \frac{d u}{d x}=2 x \\
& \frac{d y}{d x}=\frac{d y}{d u} \cdot \frac{d u}{d x} \\
& \frac{d y}{d x}=\cos (u) \cdot 2 x \\
& \frac{d u}{d x}=2 x \cos \left(x^{2}\right)
\end{aligned}
$$

Thanks to the quotient rule, it is relatively easy to find the derivative of the tangent function.
The quotient rule says that for $y=\frac{u}{v^{\prime}} \frac{d y}{d x}=\frac{v \frac{d u}{d x}-u \frac{d v}{d x}}{v^{2}}$. Here $u$ is $\sin x$ and $v$ is $\cos x$. Remember that the derivative of $\sin x$ is $\cos x$, and the derivative of $\cos x$ is $-\sin x$.
$f(x)=\tan x$
$f(x)=\frac{\sin x}{\cos x} \quad$ Now take the derivative:
$f^{\prime}(x)=\frac{\cos x \cos x-(\sin x \cdot-\sin x)}{(\cos x)^{2}}$
$f^{\prime}(x)=\frac{\cos x \cos x+\sin x \sin x}{(\cos x)^{2}}$
$(\cos x)^{2}$ and $(\sin x)^{2}$ are normally written as $\cos ^{2} x$ and $\sin ^{2} x$.
$f^{\prime}(x)=\frac{\cos ^{2} x+\sin ^{2} x}{\cos ^{2} x}$
From trigonometry, you should remember that $\cos ^{2} x+\sin ^{2} x=1$ :
$f^{\prime}(x)=\frac{1}{\cos ^{2} x}$
Since $\frac{1}{\cos x}=\sec x$, the final form of the derivative is:
$f^{\prime}(x)=\sec ^{2} x$

You can see that the derivative of the tangent is the secant squared.

## Practice

Show that the derivative of $\cot x$ is $-\csc ^{2} x$. Use the fact that $-\sin ^{2} x-\cos ^{2} x=-\left(\cos ^{2} x+\sin ^{2} x\right)=-1$.

## Example

Find $\lim _{h \rightarrow 0} \frac{\sin \left(\frac{\pi}{2}+h\right)-1}{h}$.
This limit is a derivative, and more specifically it is a derivative involving the function $f(x)=\sin x$. If you write the derivative of this function in a general way, it looks like this:
$\frac{d y}{d x}=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}=\lim _{h \rightarrow 0} \frac{\sin (x+h)-\sin x}{h}$
Notice that we are finding the infinitely small change in $y$ by taking the value of the function at $x+h$, and then subtracting the function value at $x$. When $h$ is infinitely small, it is equal to $d x$, the infinitely tiny change in $x$.

When you look at the problem, you see that there is no $x$. Instead, this expression refers to the derivative of $\sin x$ at $x=\frac{\pi}{2}$. The derivative of $\sin x$ is $\cos x$, and when $x$ is $\frac{\pi}{2}$ the value of the derivative is $\cos \frac{\pi}{2}=0$.

## V. Applying Derivatives

## Maxima, Minima, and Optimization

A maximum or minimum has to be located at a critical point: $f^{\prime}(x)=0$ or $f^{\prime}(x)$ does not exist.
Caution - a critical point may not be a minimum or maximum. Use the First Derivative Test:
If $f^{\prime}(x)$ changes from positive to negative there is a maximum.
If $f^{\prime}(x)$ changes from negative to positive there is a minimum.

No change on either side of the critical point means no minimum or maximum there.

If you use a graphing calculator or graphing software it is often quite easy to find the maximum or minimum of a function. Even if graphing calculators are not allowed for your course you can use them to check your work. The purpose of this section is to find the maxima and minima of a function by looking at its derivative.

As we saw earlier, when a function is increasing, its derivative is positive. That makes sense, since the derivative measures the rate of change, which is positive when the function is increasing. When the function is decreasing the derivative is negative. The interesting thing is what happens in between. If a function is increasing at first, and then starts to decrease, the point where the increase stops is a maximum value point. This could be the function's overall maximum, or it could be just a local maximum if the function starts to increase again somewhere else. Just before a maximum value point, the derivative is positive, and just afterwards it is negative.

If the function has a nice round top, the slope of the tangent line is actually zero right at the maximum point. The derivative is zero there.

## Maximum



The same goes for a minimum point. Here the function is decreasing at first, and then it starts to increase. The derivative is negative at first, then zero at the minimum point, and then positive as the function increases again:

## Minimum



0

Occasionally the graph of a function has a sharp point as a maximum or minimum, as in the case of the absolute value function $f(x)=|x|$. If you graph this function, you'll see that the minimum is a sharp point or "cusp" at $x=0$. Recall that we use the concept of a limit to create a tangent line. Approaching zero from the right, the slope of the line is the limit: $\lim _{x \rightarrow 0} \frac{|x|-|0|}{x-0}=$ $\lim _{x \rightarrow 0} \frac{|x|}{x}$ which is 1. Approaching it from the left however, $x$ is negative and the slope is -1 .

Since we get a different result depending on whether we approach 0 from the left or from the right, we can't create a tangent line at zero. The limit doesn't exist, which means no tangent line, and therefore no derivative at this minimum point.

We just looked at two different ways that a continuous function can have a minimum or a maximum at a certain point.

If a function has a maximum somewhere, the derivative must change from positive to negative. If the function has a minimum the derivative must change from negative to positive. A change in sign can only happen at a point where the derivative is either zero or it doesn't exist.

This is called The First Derivative Test, and we can actually find the minimum or maximum this way. Be careful though: a function can have a derivative of zero (a horizontal tangent line) somewhere where there is no minimum or maximum. To see this, create a graph of the function $f(x)=x^{3}$, magnified near $x=0$. Right at zero, the tangent line becomes horizontal, but this point is neither a minimum nor a maximum. Without a graph, you can use the derivative: $3 x^{2}$ is positive before $x=0$, and after too. In the same way, it is possible that the derivative of a function doesn't exist at a certain point, but there is neither a minimum nor a maximum there. For example, the function $f(x)=\sqrt[3]{\mathrm{x}}$ has a vertical tangent line at $\mathrm{x}=0$. A vertical tangent means no slope, because the "run" would be 0 . The derivative doesn't exist, but the graph shows no minimum or maximum at $x=0$. Also, the derivative doesn't change signs.

Anyway, this topic is fairly straightforward. To find the maximum or minimum, simply look for where the derivative changes signs. That can happen where the derivative is 0 or doesn't exist. These are the potential minimum or maximum points. Check if the derivative changes sign on either side of this potential point. If it does, you have found a maximum or minimum, although it may just be a local max or min.

To make this a little more challenging, your calculus book may ask you find the maximum or minimum of a function on a specified interval. This means that the maximum or minimum value may occur at the end points of the interval, so don't forget to check those! Simply calculate the value of the function at these points (the $y$-value), and compare that with the $y$-value at your potential minimum and maximum points.

You may see some fairly complicated fractions in your derivative so it may seem harder to figure out where it would be zero. Just keep in mind that if the top part of the fraction is zero (but the bottom is not), the whole thing is zero. If the bottom part of the fraction is zero, the derivative doesn't exist because we can't divide by zero.

## Example

A fairly standard problem for this subject involves fencing in an area to provide the maximum area with an available length $p$ of fencing. Here $p$ would be the perimeter. The fenced-in area is length times width, so $A=I w$. If we call the length and width of the fenced in area $I$ and $w$, we could say that $2 l+2 w=p$, or $I+w=1 / 2 p$. Because $p$ is just a number, it is possible to eliminate one of the variables I and $w$. Let's say that $I=1 / 2 p-w$. Now our area is $(1 / 2 p-w) w$, or $A=1 / 2 p w-w^{2}$. We are interested in how the area changes as we change $w$, and specifically where the maximum area occurs. We take the derivative $\frac{d A}{d w}$, which is $1 / 2 p-2 w$, and set it to 0 . Here we do not need to be concerned with where the derivative exists, because it exists everywhere. Solving $1 / 2 p-2 w=0$, we get $2 w=1 / 2 p$, or $w=1 / 4 p$. Since $I+w=1 / 2 p$, we can conclude that the length I would be $1 / 4 \mathrm{p}$ also, and the shape that we are fencing in is a square. This shape is either a maximum or a minimum of the area function. Notice that if $w$ is smaller than $1 / 4 p$, the derivative is positive, and when it is larger the derivative is negative. This shows that the area is increasing when $w$ is smaller, and decreasing when $w$ is bigger than the calculated value. The square shape represents the maximum area that can be fenced in with a fence of length $p$. [People who create math problems incorrectly assume that this interesting fact is well known among students. As a result you are very unlikely to ever be able to take advantage of it on a test.]

## Example

Find the point on the graph of $f(x)=2 x$ that is closest to the point $(3, o)$ on the $x$-axis.
This problem asks you to minimize the distance between two points. Just as you did before calculus, you should use the Pythagorean Theorem to express this distance: $a^{2}+b^{2}=c^{2}$. In this case c is the distance d , and we will use a for the horizontal distance and b as the vertical distance. We can minimize the distance, but it will be easier to minimize the square of the distance.
$d^{2}=\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}$
We have one of the points, but the point on the graph is unknown. Use the generic point ( $x, y$ ), which in this case will be ( $x, 2 x$ ):
$d^{2}=(x-3)^{2}+(2 x-0)^{2}$
$d^{2}=x^{2}-6 x+9+4 x^{2}=5 x^{2}-6 x+9$
The derivative is $10 x-6$, which is zero at $x=\frac{3}{5}$. The point is $(0.6,1.2)$.

## Position, Velocity, and Acceleration

Velocity is speed with a direction. It can be positive or negative.
Velocity is the derivative (rate of change) of the position, and acceleration is the derivative of the velocity.

An object is speeding up if the acceleration and the velocity have the same sign.


Photo: Patrick Ch. Apfeld

Velocity is the derivative of the position function.
The derivative measures how fast something is changing. If the position of an object changes with time the derivative gives us the speed of the object, or more precisely its velocity. Speed is always a positive quantity, just like length, but velocity can be negative. In physics we usually say that the velocity is positive if an object is moving up, and negative if an object is moving down. Mathematicians often visualize an object moving along the $x$-axis. In this case you
would consider the velocity as positive when the object is moving to the right, and negative when the object is moving to the left.

Acceleration is the derivative of the velocity function

If the velocity of an object changes, we call that acceleration. Acceleration can be either positive or negative, depending the direction it acts in. Negative acceleration doesn't necessarily mean "slowing down"! Acceleration due to gravity is usually thought of as negative, since an arrow representing it would point downwards. A ball thrown upwards will experience a constant negative acceleration. This will cause it to slow down at first, and then speed up as it reverses direction.

Velocity has a direction, so we can say that an object is speeding up if the acceleration and the velocity are in the same direction (they have the same sign). The object slows down if the velocity and acceleration have opposite signs. This is important for problems that ask you to determine if something is speeding up or slowing down in a given interval. You are likely to need an understanding of this for the AP test.

## Rolle's Theorem and The Mean Value Theorem

Rolle's Theorem says that if the average rate of change over an interval is zero, then there has to be at least one point in the interval where the derivative (the instantaneous rate of change) is zero:

If $\frac{f(b)-f(a)}{b-a}=0$, then there is some point $c,[a<c<b]$, where $f^{\prime}(c)=0$, provided that $f(x)$ is continuous on the interval $[a, b]$ and differentiable on ( $a, b$ ).

The Mean Value Theorem says that there is at least one point in an interval where the derivative is equal to the average rate of change:

There is at least one number $c$, in the interval $[a, b]$, such that $f^{\prime}(c)=\frac{f(b)-f(a)}{b-a}$, provided that $f(x)$ is continuous on the interval $[a, b]$ and differentiable on $(a, b)$.

Please be careful here. This section contains some shocking truths that could totally overwhelm you if you are not properly prepared for them. Let's have a look.

What goes up must come down, provided it doesn't reach escape velocity. You won't be surprised if a ball you throw up in the air comes back down to where you can catch it. However, here is the first shocking fact: in order to return to the same position where it first left your hand, the ball must stop going up, and start coming down! For one single point in time, the speed of the ball is actually zero. Speed is continuous (at least, we assume it is), and for the velocity of the ball to change from positive to negative it must pass through 0.

Here is a picture of this situation, with the velocity shown in red and the position superimposed in blue:


Another way of looking at this is to consider the position function, shown in blue above. I chose to call the position of the hand of the person throwing the ball as 0 when I graphed this. At the start of the interval the position is 0 , and at the end of the interval when the person catches the ball, the position is also 0 . It would make no difference if I had labeled the starting position as 1 meter and the end position as 1 meter. The net change in the position over the interval is zero. The average rate of change of the position is the net change divided by the length of the interval. This is actually the average speed (velocity in this case). The average velocity is zero. There is only one possible conclusion we can draw from this example. If the average rate of change in position over an interval (the average velocity) is zero, then there has to be at least one point in the interval where the speed is actually zero. Be sure to take some time to recover from this shocking revelation, and convince yourself that this is really a fundamental truth of the universe in which you live.

Well, all of that was really just common sense. Velocity is the rate of change of position, so it is the derivative of the position function. We can say that if a function has a net change of zero over an interval, then there must be at least one point in that interval where the derivative is zero. That is called Rolle's Theorem. It comes with some cautions that are already implied in our real-life example. First, the function must be continuous between the two points where it has the zero value. So, if the function value is zero at point $A$ and then zero again at point $B$, the function needs to be continuous over the entire interval $[A, B]$. Second, the function needs to have a derivative everywhere, except at the actual endpoints. In the example with the ball, you would probably describe the motion as a function of time. At $t=0$, the position is 0 . The function doesn't exist for $t<0$, so there is no left-hand limit at $t=0$ as required by the definition of the derivative. That is not a problem that would affect the truth of the theorem. The requirement is that the function should be differentiable on the open interval ( $\mathrm{A}, \mathrm{B}$ ).

Here is Rolle's Theorem, and it says in math language what we just said in words:

If $\frac{f(b)-f(a)}{b-a}=0$, then there is some point $c,[a<c<b]$, where $f^{\prime}(c)=0$, provided that $f(x)$ is continuous on the interval $[a, b]$ and differentiable on $(a, b)$.

If you feel well enough, we can continue on to the next example. In this example, I drive from somewhere in Orlando to some place near Tampa. These two places are exactly 120 miles apart, and by an amazing coincidence it takes me exactly 2 hours to arrive at my destination. You can conclude that my average speed during this trip was 60 miles per hour. And here is the second shocking fact: it would have been impossible for me to complete my journey without attaining an exact speed of 60 miles per hour at some moment during the trip. If I was driving at a speed of less than 60 mph for part of the journey, I would have to speed up at some point to reach a speed greater than 60 mph , so I could compensate for going more slowly for a while.

If the average rate of change of a function is 60 , then there must be at least one point where the derivative (the instantaneous rate of change) is actually 60. Again, the same conditions about continuity and differentiability apply. So, if $\frac{f(b)-f(a)}{b-a}=60$, then there is some point $c$, [ $a<c<b]$, where $f^{\prime}(c)=60$, provided that $f(x)$ is continuous on the interval $[a, b]$ and differentiable on ( $a, b$ ).

The general idea is called the Mean Value Theorem, and yes, it is just common sense ${ }^{2}$ :
If $f(x)$ is defined and continuous on the interval $[a, b]$ and differentiable on $(a, b)$, then there is at least one number $c$ such that $f^{\prime}(c)=\frac{f(b)-f(a)}{b-a}$ for $[a<c<b]$.

## Second Derivatives

The second derivative, $f^{\prime \prime}(x)$ or $\frac{d^{2} y}{d x^{2}}$, gives the rate of change of the first derivative. $f^{\prime \prime}(x)>0$ : the curve is concave up.
$\mathrm{f}^{\prime \prime}(\mathrm{x})<0$ : the curve is concave down.

Potential inflection points are located where the second derivative is zero or doesn't exist.

Velocity is the derivative of the position function. Acceleration is the derivative of the velocity, and the derivative of the derivative of the position. It is the second derivative of the position function. You may take the second derivative of any function.

Survival Tip: Homework assignments on this topic tend to be long and tedious. After you find what you think is the first derivative, check it using an online derivative calculator. That way you don't spend a long time finding the second derivative of the wrong first derivative.

The second derivative is the change of the derivative, with respect to $x$. That is, we want the infinitely small change in the first derivative, or $d\left(\frac{d y}{d x}\right)$, divided by $d x$. This may be written as $\frac{d^{2} y}{d x^{2}}$ or as $f^{\prime \prime}(x)$. For example, the second derivative of the sine function is the derivative of the cosine function, which is $-\sin x$.

Graph the function $y=x^{3}-3 x+4$ using graphing software. Enter $y=x^{\wedge} 3-3 x+4$ in the Input space. To draw a tangent line to this curve at $x=-2$, enter $y=9 x+20$. The slope of this line is 9. As you move further to the right, toward $x=-1$, the value of the slope of decreases until it is 0 for a tangent line drawn at $x=-1$. The function has a local maximum here. Further to the right the slope continues to decrease as it becomes more and more negative. The derivative is decreasing, which means that the second derivative is negative. The second derivative measures the rate of change of the first derivative. In fact, whenever a curve is concave down, the second derivative will be negative.

At $x=0$ the slope of the tangent line reaches a (local) maximum negative value, and then it begins to increase. It continues to increase, reaching 0 at $x=1$, and then becoming more and more positive. Because the value of the first derivative is increasing, the second derivative is positive. Whenever a curve is concave up, the second derivative is positive.

I find it hard to think of that each time, so I remember it like this:


Notice that at $x=0$ the curve changes from being concave down to being concave up. At this point the second derivative changes from negative to positive, so it is 0 here. A point where the concavity of a curve changes is called an inflection point. To find the location of the inflection points of a curve, look for where the second derivative changes signs. This has the potential to occur whenever the second derivative is zero or doesn't exist. However, just like for the first derivative, these points just indicate potential sign changes. For example, the second derivative of the function $y=x^{4}$ is 0 at $x=0$, but there is no inflection point at that location.

## The Second Derivative Test

If $f^{\prime}(x)=0$ and $f^{\prime \prime}(x)$ is positive, there is a minimum at that point.
If $f^{\prime}(x)=0$ and $f^{\prime \prime}(x)$ is negative, the point is a maximum.
If $\mathrm{f}^{\prime \prime}(\mathrm{x})$ is zero or doesn't exist, the Second Derivative Test is inconclusive. Use the First Derivative Test.

The second derivative can also help us determine whether a point is a minimum or a maximum. If a function has a local or global maximum at a point where it is differentiable, the first
derivative is 0 . If the second derivative is negative at this point, the curve will be concave down which means that there is a maximum here:


If the second derivative is positive we conclude that the curve is concave up so we must have a minimum:


However, there are times when the second derivative is 0 at the point in question so that gives us no information. The second derivative may also be hard to find. In such cases look at the first derivative to see whether it changes from positive to negative (a maximum), or from negative to positive (a minimum).

## Implicit Differentiation

Treat y as a function of x , even if it is not stated that way, e.g. $\mathrm{x}^{2}+\mathrm{y}^{2}=100$.
$y^{2}$ is composed of an outer function and an inner function. Take the derivative as $2 y \cdot \frac{d y}{d x}$ using the Chain Rule.
$x^{2}+y^{2}=100 \rightarrow 2 x+2 y y^{\prime}=0 \rightarrow 2 y y^{\prime}=-2 x \rightarrow y^{\prime}=-\frac{x}{y}= \pm \frac{x}{\sqrt{100-x^{2}}}$

When I first learned about implicit differentiation I found it very confusing. It was not so hard to learn what to do, but it wasn't clear why we were doing it that way.

Implicit differentiation is used when it is not convenient, or not possible, to write a function or relation in the form $y=\ldots$. As an example, we will use the equation $x^{2}+y^{2}=100$. It is a bit inconvenient to rewrite that as: $y= \pm \sqrt{100-x^{2}}$, and even more inconvenient to get the derivative so we could construct a tangent line to the circle. Still, we can do that by writing
$y=\left(100-x^{2}\right)^{1 / 2}$ and $y=-\left(100-x^{2}\right)^{1 / 2}$
Now use the chain rule to get $y=\frac{1}{2}\left(100-x^{2}\right)^{-1 / 2} \cdot-2 x$ and $y=-\frac{1}{2}\left(100-x^{2}\right)^{-1 / 2} \cdot-2 x$. Once you rewrite that nicely you have $y= \pm \frac{x}{\sqrt{100-x^{2}}}$.

Fortunately there is an much easier way.
If you have read the short e-book "What is Calculus - A Bedtime Story", you will want to just use differentials here:
$x^{2}+y^{2}=100$
$2 x d x+2 y d y=0 \quad$ Divide both sides by $d x:$
$2 x \frac{d x}{d x}+2 y \frac{d y}{d x}=0$
$2 x+2 y \frac{d y}{d x}=0$
$2 y \frac{d y}{d x}=-2 x$
$\frac{d y}{d x}=-\frac{2 x}{2 y}$
$\frac{d y}{d x}=-\frac{x}{y}$
Substitute $y= \pm \sqrt{100-x^{2}}: \frac{d y}{d x}= \pm \frac{x}{\sqrt{100-x^{2}}}$

When we use implicit differentiation we "imply" that $y$ is a function of $x$, but we just leave the equation as it is:
$x^{2}+y^{2}=100$

Using the implicit assumption that $y$ is a function of $x$, you can differentiate $y^{2}+x^{2}=100$ directly using the chain rule. Notice that $\mathrm{y}^{2}$ has an outer function, the squared part (with a derivative of $2 y$ ), and an inner function $y(x)$. The derivative of $y(x)$ is simply $\frac{d y}{d x}$. So, taking the derivative of the first term, $y^{2}$, gives us $2 y \cdot \frac{d y}{d x}$. To get the derivative of $x^{2}$, with respect to $x$, take the derivative of the outer function and multiply by the derivative of the inner function:
$2 x \cdot \frac{\mathrm{dx}}{\mathrm{dx}}=2 \mathrm{x}$.
Differentiating $y^{2}+x^{2}=100$ implicitly, we get:
$2 y \frac{d y}{d x}+2 x=0$
Next, divide both sides by 2 :
$y \frac{d y}{d x}+x=0$
Now the derivative we want is right there:
$y \frac{d y}{d x}=-x$
$\frac{d y}{d x}=-\frac{x}{y} \quad$ Substitute $y= \pm \sqrt{100-x^{2}}$ :
$\frac{d y}{d x}= \pm \frac{x}{\sqrt{100-x^{2}}}$
Because $y^{2}+x^{2}=100$ is a circle, the slope of the tangent line at point $(x, y)$ may be positive or negative depending on your choice of $x$ and $y$.

Stop here for a moment to appreciate the fact that if it is too difficult to solve a particular equation for y you may be stuck with an expression like $\frac{d y}{d x}=-\frac{x}{y}$. You will see this type of expression again when we look at differential equations.

## Example 1

Find the point(s), if any, where the graph of $y^{2}+x^{2}=100$ has a horizontal tangent line.
Since we already know that $\frac{d y}{d x}=-\frac{x}{y}$ here, all we have to do is find where $-\frac{x}{y}=0$. If you think about it for a bit, or multiply both sides by y , you'll realize that the only way that $-\frac{\mathrm{x}}{\mathrm{y}}$ can be zero is if the top part of the fraction, $x$, is equal to zero. On the other hand, $y$ must not be equal to zero, because a division by 0 would mean that the derivative does not exist. If you find what you think is a suitable value for $x$, go back and check that the resulting value for $y$ actually allows the derivative to be zero!

There are only two potential solutions. When $x=0$, the original equation tells us that $y^{2}=100$, which means that $y=10$ or -10 . The two points are $(0,10)$ and $(0,-10)$. Checking the derivative at both these points gives us
$-\frac{x}{y}=-\frac{0}{10}=0$ and
$-\frac{x}{y}=-\frac{0}{-10}=0$.
The graph has horizontal tangent lines at the top and the bottom of the circle, which is as we would expect.

## Example 2

If $y^{3}+x^{3}-10 x y=0$, find $d y / d x$.

Now the only way to find the derivative is through implicit differentiation. We have to use both the chain rule and the product rule:
$3 y^{2} \cdot \frac{d y}{d x}+3 x^{2}-10\left(1 \cdot y+\frac{d y}{d x} \cdot x\right)=0$. Remember that the derivative of a constant is always 0 , even if that constant is 0 !

It is a bit of a pain to write out the fraction $\frac{d y}{d x}$ as you do your calculation, so many people use the shorthand notation $y^{\prime}$ here instead:
$3 y^{2} y^{\prime}+3 x^{2}-10\left(y+x y^{\prime}\right)=0$
$3 y^{2} y^{\prime}+3 x^{2}-10 y-10 x y^{\prime}=0$
Although this looks a bit hard, equations of this kind can always be solved for $y^{\prime}$, the derivative we want, by moving all of the terms containing $y^{\prime}$ to one side of the equation:
$3 y^{2} y^{\prime}-10 x y^{\prime}=10 y-3 x^{2}$
$y^{\prime}\left(3 y^{2}-10 x\right)=10 y-3 x^{2}$
$y^{\prime}=\frac{10 y-3 x^{2}}{3 y^{2}-10 x}$
$\frac{d y}{d x}=\frac{10 y-3 x^{2}}{3 y^{2}-10 x}$
Because we can't solve the original equation for $y$, the expression for the derivative can't be written only in terms of $x$.

## Example 3

If $f(x)+g(y)=3 x+2 y^{2}$, find $\frac{d y}{d x}$.
Even though there are unspecified functions $f$ and $g$, we can take the derivative in a general way. Note however that for $g(y), y$ is still considered a function of $x$, so treat $y$ as the "inner" function:
$f^{\prime}(x)+g^{\prime}(y) y^{\prime}=3+4 y y^{\prime}$
$g^{\prime}(y) y^{\prime}=-f^{\prime}(x)+3+4 y y^{\prime}$
$g^{\prime}(y) y^{\prime}-4 y y^{\prime}=3-f^{\prime}(x)$
$y^{\prime}\left(g^{\prime}(y)-4 y\right)=3-f^{\prime}(x)$
$y^{\prime}=\frac{3-f^{\prime}(x)}{g^{\prime}(y)-4 y}$

## Related Rates

For Related Rates problems, there is always a formula that relates one changing quantity to another. Implied in this is that both quantities are functions of time. Differentiate both sides of the equation (implicitly) with respect to time:
$\mathrm{C}=\pi \mathrm{r}^{2} \rightarrow \frac{\mathrm{dC}}{\mathrm{dt}}=\pi \cdot 2 \mathrm{r} \frac{\mathrm{dr}}{\mathrm{dt}}$

Many times the change in one thing depends on the change in something else. For example, the circumference of a circle changes as we change the radius. However, the change in the radius may also depend on something, like maybe time. Consider the following example:

Bob likes to keep his lawn looking just perfect. One day he is horrified to discover that a fairy ring fungus has invaded his beautiful turf, creating an ugly circle of mushrooms. In spite of Bob's efforts to control the fungus, the radius of the ring continues to increase at a steady rate of 3 inches a day. How fast is the circumference (the part with the mushrooms) increasing?


Fairy Ring Fungus

Obviously, the circumference of the ring depends on the radius, which is increasing with time.
$C$ is just a function of $r: C(r)=2 \pi r$. Also, $r$ is changing as a function of time: $r(t)=3 t$ (inches).
Let's make a little table to describe this situation:

| $\mathbf{t}$ (days) | $\mathbf{r}$ (inches) | Circumference <br> (inches) |
| :---: | :---: | :---: |
| 1 | 3 | $6 \pi$ |
| 2 | 6 | $12 \pi$ |
| 3 | 9 | $18 \pi$ |
| 4 | 12 | $24 \pi$ |

Calculus tells us that since $C=2 \pi r, \frac{\mathrm{dC}}{\mathrm{dr}}=2 \pi$.
The table shows that the circumference is in fact increasing by $2 \pi$ for every inch of increase in $r$, as expected.

We can also determine that since $r=3 t, \frac{d r}{d t}=3$

The rate of change of the circumference with respect to the radius is $2 \pi$ ( $2 \pi$ units per unit increase in $r$ ), and the radius is changing at a rate of 3 with respect to time. To get the rate of change of something that in turn depends on the rate of change of something else, you can use the Chain Rule and multiply the two rates of change:
$\frac{\mathrm{dC}}{\mathrm{dt}}=\frac{\mathrm{dC}}{\mathrm{dr}} \cdot \frac{\mathrm{dr}}{\mathrm{dt}}=2 \pi \cdot 3=6 \pi$
You can see that dr cancels out to give you the derivative you want, $\frac{\mathrm{dC}}{\mathrm{dt}}$. Check the table to confirm that the rate of change of the circumference is $6 \pi$ inches per day.

By now you know implicit differentiation, so you should be able to get $\frac{\mathrm{dC}}{\mathrm{dt}}$ directly. Implied in the equation $C=2 \pi r$ for this problem is that both $C$ and $r$ are functions of the time $t$.
Differentiate both sides with respect to $t$ :
$C=2 \pi r$
$\frac{d C}{d t}=2 \pi \cdot \frac{d r}{d t}$
$\frac{\mathrm{dC}}{\mathrm{dt}}=2 \pi \cdot 3=6 \pi$
This method gives the same result, and it is a lot easier to use for more complicated situations. Try it out for yourself by using a simple example like the area of an expanding square.

Related rates problems take a basic principle like that of the fairy ring fungus, and create endless variations of it. The simplest examples may involve expanding balloons or melting snowballs, while the more complex ones have elaborate descriptions involving angles. However, they all have something in common, which is that there is a formula that relates one changing quantity to another changing quantity. In the case of the fungus, the changing circumference could be linked to the changing radius by the simple formula $C=2 \pi r$. We were able to differentiate both sides with respect to $t$ to get the derivative $\frac{\mathrm{dC}}{\mathrm{dt}}$ which was $6 \pi$. Note that this is a nice constant rate of change that doesn't depend on the value of $r$. Authors of calculus textbooks usually go to a lot of trouble to provide more interesting problems for you that do have a varying rate of change.

## Example 1

Here is a real classic: the sliding ladder problem.

A 10 foot ladder is leaning against a wall, but it starts to slide down, at a rate of $2 \mathrm{feet} / \mathrm{sec}$. How fast is the bottom of the ladder moving away from the wall when the top of the ladder is 6 feet from the ground?

First we need an equation that relates the two distances involved (the distance along the wall from the top of the ladder to the ground, and the distance between the bottom of the ladder and the wall). Always draw yourself a nice picture of the situation. In this case you'll see that the two distances are related through the Pythagorean Theorem. If we call the distance along the wall $a$, and the bottom distance $b$, we can write $a^{2}+b^{2}=100$. (Hmm, that looks suspiciously like the equation we used in the implicit differentiation section.) Both a and bare changing, and the problem tells us the rate of change of a. The ladder is sliding down the wall at a rate of $2 \mathrm{ft} / \mathrm{sec}$, meaning distance a is decreasing at a rate of $2 \mathrm{ft} / \mathrm{sec}$. Because the distance is decreasing rather than increasing we say that the rate of change is negative: $\frac{\mathrm{da}}{\mathrm{dt}}=-2$.

I usually put a "have" and "want" on my paper. In this case I would write: Have: $\frac{\mathrm{da}}{\mathrm{dt}}$ Want: $\frac{\mathrm{db}}{\mathrm{dt}}$ Start by writing the equation that relates the two distances a and b :
$a^{2}+b^{2}=100$
Although the equation doesn't explicitly say so, both $a$ and $b$ are changing with time so they are functions of $t$. Take the derivative with respect to $t$ on both sides of the equation:
$2 \mathrm{a} \frac{\mathrm{da}}{\mathrm{dt}}+2 \mathrm{~b} \frac{\mathrm{db}}{\mathrm{dt}}=0$ (Notice that the constant 100 doesn't change with time, so its derivative is 0 )
This is where I look at my "have and want" statement. I want $\frac{\mathrm{db}}{\mathrm{dt}}$, and I can fill in $\frac{\mathrm{da}}{\mathrm{dt}}$.
You can substitute that directly, or clean the equation up a bit first by dividing both sides by 2 :
$2 a \frac{d a}{d t}+2 b \frac{d b}{d t}=0$
$a \frac{d a}{d t}+b \frac{d b}{d t}=0$
$a(-2)+b \frac{d b}{d t}=0$
$-2 a=-b \frac{d b}{d t}$
$\frac{\mathrm{db}}{\mathrm{dt}}=\frac{2 \mathrm{a}}{\mathrm{b}}$

This tells us that the rate of change of $b$ varies depending on the ratio of $a$ to $b$. We have $a$ general equation that describes how fast $b$ is changing, but we only get a specific rate if we pick a specific point in time. The problem must supply this information, so it asks for the rate when the top of the ladder is 6 feet from the ground. When $a$ is $6, b$ is 8 (since $a^{2}+b^{2}=100$ ), and $\frac{\mathrm{db}}{\mathrm{dt}}$ is $\frac{2 \cdot 6}{8}=1.5$ feet $/ \mathrm{sec}$. This rate of change is positive, since the distance b increases as the ladder slides down the wall.

## Example 2

The volume of a cube is increasing at a steady rate of $150 \mathrm{in}^{3}$ per hour. How fast is the surface area increasing when the sides of the cube are 10 inches?

We have $\frac{\mathrm{dV}}{\mathrm{dt}}$, and we want $\frac{\mathrm{dA}}{\mathrm{dt}}$. The rate of change will likely not be constant, because the problem asks for $\frac{\mathrm{dA}}{\mathrm{dt}}$ at a specific point: "when the sides of the cube are 10 inches."

In this variation of related rates, there are two separate equations that relate the three quantities (volume, area, sides):
$\mathrm{V}=\mathrm{s}^{3}$ and $\mathrm{A}=6 \mathrm{~s}^{2}$
There are several ways to solve this problem. One way is to differentiate both equations separately.
$V=s^{3}$
$\frac{d V}{d t}=3 s^{2} \frac{d s}{d t}$
At the point where $\mathrm{s}=10: 150=3 \cdot 10^{2} \frac{\mathrm{ds}}{\mathrm{dt}}$, so $\frac{\mathrm{ds}}{\mathrm{dt}}=0.5$ inches per hour.
$A=6 s^{2}$
$\frac{\mathrm{dA}}{\mathrm{dt}}=12 \mathrm{~s} \frac{\mathrm{ds}}{\mathrm{dt}}$
At the point where $\mathrm{s}=10: \quad \frac{\mathrm{dA}}{\mathrm{dt}}=12 \cdot 10 \cdot 0.5=60$ square inches per hour.

We can also get the answer by combining the two equations. If $V=s^{3}$, then $s=V^{1 / 3}$. Substitute that into $A=6 s^{2}$ to get $A=6 V^{2 / 3}$. Now take the derivative of this last equation with respect to time:
$\frac{\mathrm{dA}}{\mathrm{dt}}=6 \cdot \frac{2}{3} \mathrm{~V}^{-1 / 3} \frac{\mathrm{dV}}{\mathrm{dt}}$
When the sides are 10 inches, the volume is $1000 \mathrm{in}^{3}$ and $\mathrm{V}^{-1 / 3}=\frac{1}{\sqrt[3]{1000}}=\frac{1}{10}$
$\frac{\mathrm{dA}}{\mathrm{dt}}=4 \cdot \frac{1}{10} \cdot 150=60 \mathrm{in}^{2} / \mathrm{hr}$.
You can combine the equations the other way too: If $A=6 s^{2}$ then $s=\left(\frac{A}{6}\right)^{\frac{1}{2}} . V=s^{3}=\left(\frac{A}{6}\right)^{\frac{3}{2}}$.
Remember to use the chain rule to differentiate $V=\left(\frac{A}{6}\right)^{\frac{3}{2}}$, since $A$ is a function of $t$ :
$\frac{\mathrm{dV}}{\mathrm{dt}}=\frac{3}{2}\left(\frac{\mathrm{~A}}{6}\right)^{\frac{1}{2}} \cdot \frac{1}{6} \frac{\mathrm{dA}}{\mathrm{dt}}$
When the sides are 10 inches, the total surface area $A$ is $600 \mathrm{in}^{2}: \frac{\mathrm{dV}}{\mathrm{dt}}=\frac{3}{2}\left(\frac{600}{6}\right)^{\frac{1}{2}} \cdot \frac{1}{6} \frac{\mathrm{dA}}{\mathrm{dt}}$, so $150=\frac{3}{2} \cdot 10 \cdot \frac{1}{6} \frac{\mathrm{dA}}{\mathrm{dt}}$, and $\frac{\mathrm{dA}}{\mathrm{dt}}=60$.

## Example 3

Sometimes one of the changing quantities is an angle. Always work with the angle in radians. The rate of change of the angle will be in radians per unit of time, such as radians/sec.

Lucia is marking the spot in her yard where she wants to put her new shed. She places a stake at the southwest corner and ties two strings to it. Then she walks directly north 10 feet and attaches the first string to a stake at the northwest corner. The northeast corner will be 15 feet from the northwest corner, and Lucia is using the second string to check that her angle is straight. [According to the Pythagorean Theorem, if the angle is straight then the length of the second string will be 18 feet and $1 / 3$ inch when it stretches from the southwest to the northeast corner.] Starting at the northwest corner, Lucia walks to the northeast corner at a rate of 4 feet per second, while holding the second string taut. How fast is the angle between the two strings changing when she is halfway between the two corners?


Have: $\frac{\mathrm{dx}}{\mathrm{dt}}=4 \mathrm{ft} / \mathrm{sec}$ Want: $\frac{\mathrm{d} \theta}{\mathrm{dt}}$
Be careful here so you don't end up with more than two unknowns in your equation. We could say that $\sin \theta=x / y$, but then we have three unknowns. It is possible to express $y$ in terms of $x$ by using the Pythagorean Theorem, but the resulting equation is complex and hard to differentiate. The best way to relate the angle $\theta$ to the distance x is by using the tangent:
$\tan \theta=\frac{\mathrm{x}}{10}$. Now differentiate both sides with respect to t :
$\sec ^{2} \theta \frac{\mathrm{~d} \theta}{\mathrm{dt}}=\frac{1}{10} \frac{\mathrm{dx}}{\mathrm{dt}}$.
$\frac{1}{\cos ^{2} \theta} \frac{\mathrm{~d} \theta}{\mathrm{dt}}=\frac{4}{10}$
$\frac{\mathrm{d} \theta}{\mathrm{dt}}=\frac{4 \cos ^{2} \theta}{10}=\frac{2}{5} \cos ^{2} \theta$
When x is 7.5 feet, y is $\sqrt{10^{2}+7.5^{2}}=12.5$ so $\cos \theta$ is $\frac{10}{12.5}=\frac{20}{25}=\frac{4}{5}$. Squaring that we get $\frac{16}{25^{\prime}}$, so $\frac{\mathrm{d} \theta}{\mathrm{dt}}=\frac{2}{5} \cdot \frac{16}{25}=32 / 125=0.256$ radians per second.

## Example 4

A box pictured on a computer screen is increasing in size. The box is getting taller at a rate of 1 " per minute, the length is growing at a rate of 3 " per minute, and the width is increasing at a rate of 2 " per minute. How fast is the volume of the box increasing when the length is 5 ", the width is 2 " and the height is 2 "?

Have: $\frac{\mathrm{dh}}{\mathrm{dt}}, \frac{\mathrm{dl}}{\mathrm{dt}}, \frac{\mathrm{dw}}{\mathrm{dt}} \quad$ Want: $\frac{\mathrm{dV}}{\mathrm{dt}}$
The equation that relates all three quantities is $\mathrm{V}=\mathrm{l} w h$. Here $\mathrm{I}, \mathrm{w}$, and h are all functions of time, so we must use the product rule. Since the product rule deals with 2 functions rather than 3 , separate the equation: $\mathrm{V}=\mathrm{lw} \cdot \mathrm{h}$. (We are really using the product rule twice.) Now differentiate with respect to time:
$\frac{d V}{d t}=(l w)^{\prime} h+l w h^{\prime}$
$\frac{d V}{d t}=\left(\frac{d l}{d t} w+\frac{d w}{d t} l\right) h+l w \frac{d h}{d t}$
$\frac{\mathrm{dV}}{\mathrm{dt}}=\mathrm{hw} \frac{\mathrm{dl}}{\mathrm{dt}}+\mathrm{hl} \frac{\mathrm{dw}}{\mathrm{dt}}+\mathrm{lw} \frac{\mathrm{dh}}{\mathrm{dt}}$
$\frac{\mathrm{dV}}{\mathrm{dt}}=2 \cdot 2 \cdot 3+2 \cdot 5 \cdot 2+5 \cdot 2 \cdot 1=42 \mathrm{in}^{3} / \mathrm{min}$

## L'Hospital's Rule

$$
\text { If } h(x)=\frac{f(x)}{g(x)} \text { has a value of } \frac{0}{0} \text { at } x=a \text {, then } \lim _{x \rightarrow a} h(x)=\frac{f^{\prime}(x)}{g^{\prime}(x)} \text {. }
$$

Caution: Take the derivatives of the top and bottom separately, don't try to use the quotient rule!

You can use L'Hospital's Rule for limits of the form $\frac{0}{0}$ and $\frac{ \pm \infty}{ \pm \infty}$.

Suppose you are trying to draw the graph of $f(x)=\frac{\sin x}{x}$, without using a graphing calculator or graphing program. You would have to find both the top and the bottom values for different values of $x$. Let's do that now for a few values. Our top function is $y_{1}=\sin x$. Remember to work in radians, so for $x=2$ we get $y_{1}=\sin (2) \approx .9093$. The bottom function is $y_{2}=x$, so that part is easy. When $x=2, f(x) \approx \frac{.9093}{2} \approx .45$. Now try $x=1$ : $y_{1}=\sin (1)=.8415$ and $y_{2}=1$. $\mathrm{f}(\mathrm{x})=\frac{.8415}{1} \approx .84$. When $\mathrm{x}=0 \ldots$...oops, not so good. Zero over zero is considered to be an indeterminate form. The actual value of the limit may be zero, or some other number, or it may not exist at all.

To graph $f(x)$ properly, you might try calculating some values of the function close to 0 , like at $x$ $=0.1$. $\mathrm{f}(\mathrm{x})=\frac{\sin (.1)}{.1} \approx \frac{.0998}{.1} \approx .998$. Getting even closer to 0 , we try $\mathrm{x}=0.01: \mathrm{f}(\mathrm{x})=\frac{\sin (.01)}{.01} \approx \frac{.00999}{.01} \approx$ .999. Notice that both $\mathrm{y}_{1}$ and $\mathrm{y}_{2}$ get smaller as expected, since both the top and the bottom functions have a value of 0 when $x=0$. To get as close as possible to the actual value of $f(x)=$ $\frac{\sin x}{x}$ at $x=0$ we should take both the top and the bottom function at $x=0$ and then move an infinitely small distance $x$ to the right so we get an infinitely small distance $y$ as the function value. Infinitely small $x$ and $y$ distances?? Hey, that sounds like a job for ... Derivative Man! Johann Bernoulli (1667-1748) was the first to use derivatives in this situation. Unfortunately for him, he had (probably inadvertently) sold the copyright of his calculus discoveries to Marquis de L'Hospital, so this method became known as L'Hospital's Rule. Just a side note here: The Marquis de L'Hospital was a wealthy man who could have bought all kinds of things. He chose to spend a huge amount of money on what really mattered to him - calculus. That is hard to imagine when you are sitting in a calculus class watching the clock move slowly, but calculus really was the most important discovery of L'Hospital's time.

Getting back to the problem of finding the limit as $x$ goes to 0 of $f(x)=\frac{\sin x}{x}$, we want the infinitely small function value of $y_{1}=\sin x$ very close to 0 , divided by the infinitely small function value of $y_{2}=x$ very close to 0 . Since I can't actually draw infinitely small lines I have illustrated the situation by indicating $\Delta y$, which is a small increase in the function value $y$ as you move a small distance $\Delta x$ to the right of 0 :


It works the same for the bottom function, $\mathrm{y}_{2}=\mathrm{x}$ :


The function value for $f(x)=\frac{\sin x}{x}$ at a small distance $\Delta x$ to the right of 0 is $\frac{\Delta y_{1}}{\Delta y_{2}}$. Now we need the same thing for an $x$ value ( $d x$ ) infinitely close to $0 . \frac{d y_{1}}{d y_{2}}$ would do the job, and we can get that by dividing $\frac{\frac{d y_{1}}{d x}}{\frac{d y_{2}}{d x}}$.

Because we are using the same x for both functions, dx is also the same for both the top and bottom function. Now dx conveniently cancels out here to give us what we want.

We conclude that we can find the limit as $x \rightarrow 0$ of $f(x)=\frac{\sin x}{x}$ by taking the derivative of both the top and the bottom function: $\frac{\mathrm{y}_{1}{ }^{\prime}}{\mathrm{y}_{2} \prime}=\frac{\cos \mathrm{x}}{1}$. When $\mathrm{x}=0$ that value is 1 .

This was a very interesting discovery. Notice that L'Hospital's Rule was designed for the specific situation we just looked at. There is one function, divided by another function, and both functions have a value of 0 at the spot where we're trying to find the limit. That spot could be anywhere. I picked $x=0$ in the example above, but we can change that easily. $\operatorname{Try} \lim _{x \rightarrow 2} \frac{\sin (x-2)}{x-2}$. What doesn't work is to try using L'Hospital's Rule when both function values are not 0 . In that case one or both of the functions would have a large $y$ value that could not be replaced by dy like we just did.

So, at first L'Hospital's Rule was used only for limits involving the indeterminate form $\frac{0}{0}$.
L'Hospital's Rule says that we can get this limit that might otherwise elude us by taking the ratio of the derivatives. That means that we are actually taking the ratio of the rate of change of the two functions. Be careful here: take the derivatives of the top and bottom separately, don't try to use the quotient rule!

Another situation where we can't easily find a limit is if there are two functions that both approach infinity. For example, we may be trying to find the limit as $x \rightarrow \infty$ of $\frac{\ln x}{x}$.
The values of both the top and the bottom function go to infinity as x gets larger and larger.
The limit looks like $\frac{\infty}{\infty}$.
As it turns out, we can actually use L'Hospital's Rule here too. The reason for that is that as a function approaches infinity, its reciprocal approaches zero. Simply take the derivative of both functions (separately), and then find the limit as $x$ goes to infinity:
limit of $x \rightarrow \infty$ of $\frac{\ln x}{x}=\frac{\frac{1}{x}}{1}=\frac{0}{1}=0$
Using L'Hospital's Rule often involves rearranging indeterminate limits into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so you can apply the rule. Notice that if you have a product of two functions $f$ and $g$, and $f \cdot g$ works out to be $0 \cdot \infty$ in the limit, you can rearrange that as $\frac{\mathrm{f}}{\frac{1}{\mathrm{~g}}}$ to get $\frac{0}{0}$, or $\frac{\mathrm{g}}{\frac{1}{\mathrm{f}}}$ to get $\frac{\infty}{\infty}$.

## VI. Inverse Functions

## Derivatives of Exponential Functions

The function $\mathrm{y}=\mathrm{e}^{\mathrm{x}}$ increases at a rate that is equal to its value at any given point.
The derivative of $\mathrm{e}^{\mathrm{x}}$ is $\mathrm{e}^{\mathrm{x}}$.
The derivative of $a^{x}$ is $a^{x} \ln a$.

For exponential functions, the rate of increase at a particular point depends on the value of the function at that point. For example, look at the function $y=2^{x}$. When $x=1$ the value of the function (the $y$ value) is 2 . Because the function increases faster and faster, we want to approximate the rate of increase by looking at the value of the function between $x=1$ and a point very close to that, say $x=1.0001$. At $x=1.0001, y=2^{1.0001} \approx 2.000139$. The function value has increased by about 0.000139 . That means that $y$ has increased by 0.00139 units for an increase of 0.0001 units in $x$. The rate of increase is defined as the change in $y$ divided by the change in $x$, or $0.000139 \div 0.0001$ which works out to 1.39 . When $x=3, y=8$, and at $x=$ $3.0001 \mathrm{y} \approx 8.000555$. Now the rate of increase is approximately $(8.000555-8) \div 0.0001=5.55$ So, when the value of the function is 2 , the rate of increase is 1.39 , and when the value of the function is 8 , the rate of increase is 5.55 . The function $y=2^{x}$ increases at a rate that is less than its value, at any given point. If you do the same calculations for the function $y=3^{x}$, you will find that at any given point the function is increasing faster than its value at that point:

| $\mathbf{x}$ | $\mathbf{y = \mathbf { 2 } ^ { \mathbf { x } }}$ | Rate of <br> increase |  | $\mathbf{y = \mathbf { 3 } ^ { \mathbf { x } }}$ | Rate of <br> increase |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 2 | 1.39 |  | 3 | 3.30 |
|  |  |  | 3.00033 |  |  |
| 2 | 4 |  |  |  |  |


| 2.0001 | 4.000277 | 2.77 | 9.000989 | 9.89 |
| :---: | :---: | :---: | :---: | :---: |
| 3 | 8 | 5.55 | 27 | 29.7 |
| 3.0001 | 8.000555 |  | 27.00297 |  |
| 4 | 16 | 11.1 | 81 | 89.0 |
| 4.0001 | 16.00111 |  | 81.00890 |  |

There is a base, a number between 2 and 3 , for which the function $y=b^{x}$ increases at a rate exactly equal to its value at any point. This special number is the number e. Because the rate of increase of $e^{x}$ is always equal to $e^{x}$, the derivative of $e^{x}$ is just $e^{x}$.

Like $\pi$, e has an infinite number of digits. Many calculators include it as a special key that you can press to see that its value is 2.7182818228 ....

The table below shows the rate of increase for the three functions $y=2^{x}, y=e^{x}$, and $y=3^{x}$. Notice that at $\mathrm{x}=1$, the value of the function $\mathrm{y}=\mathrm{e}^{\mathrm{x}}$ is approximately 2.718 , and it is increasing at a rate of approximately 2.718 . When $x=2$, the function value is 7.389 , and the rate of increase is also 7.389 , and so on.

| x | $\mathrm{y}=\mathbf{2}^{\text {x }}$ | Rate of increase | $y=e^{x}$ | Rate of increase | $y=3^{x}$ | Rate of increase |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 2 | 1.39 | 2.718282 | 2.718 | 3 | 3.30 |
| 1.0001 | 2.000139 |  | 2.718554 |  | 3.00033 |  |
| 2 | 4 | 2.77 | 7.389056 | 7.389 | 9 | 9.89 |
| 2.0001 | 4.000277 |  | 7.389795 |  | 9.000989 |  |
| 3 | 8 | 5.55 | 20.08554 | 20.09 | 27 | 29.7 |
| 3.0001 | 8.000555 |  | 20.0876 |  | 27.00297 |  |
| 4 | 16 | 11.1 | 54.59815 | 54.60 | 81 | 89.0 |
| 4.0001 | 16.00111 |  | 54.60361 |  | 81.00890 |  |

The next image shows the graphs of $\mathrm{y}=2^{\mathrm{x}}$ (blue), $\mathrm{y}=\mathrm{e}^{\mathrm{x}}$ (green), and $\mathrm{y}=3^{\mathrm{x}}(\mathrm{red})$ :


If $\mathrm{y}=\mathrm{e}^{\mathrm{x}}$ then $\mathrm{y}^{\prime}=\mathrm{e}^{\mathrm{x}}$
Now, think about the function $y=3^{x}$. You may be very tempted to write the derivative of this function as $x \cdot 3^{x-1}$, but if you go back and look at how we developed that rule about the exponent you will see that it really doesn't apply here. For an exponential function x is the exponent, so $y=3^{x}$ is very different from $y=x^{3}$. Instead, it is very closely related to the natural exponential function $y=e^{x}$. We can rewrite $y=3^{x}$ so it uses $e$ as a base. If you raise $e$ to the power $\ln \mathrm{x}$, what you are doing is taking the natural $\log$ of x , and then putting the result into the inverse function $\mathrm{e}^{\mathrm{x}}$. Well, if you do that you just get x . This is how inverse functions work, since the inverse function "undoes" the operation of the original function. $e^{\ln x}$ is really just $x$. Grab a calculator and try it out! 3 is really the same as $\mathrm{e}^{\ln 3}$, so just write it like that:
$y=3^{x}$
$y=\left(e^{\ln 3}\right)^{x}$
$y=e^{x \ln 3}$
[Alternatively, you can take the natural $\log$ on both sides of the equation to get $\ln y=x \ln 3$, which you can then change to $e^{\ln y}=e^{x \ln 3}$ ]

When you go to differentiate $y=e^{x \ln 3}$, remember that $\ln 3$ is just a number. Use the chain rule:
$y^{\prime}=e^{x \ln 3 \cdot} \cdot \ln 3$
$y^{\prime}=\left(e^{\ln 3}\right)^{x} \ln 3 \quad$ and $e^{\ln 3}$ is just 3 :
$y^{\prime}=3^{x} \ln 3$
In general, the derivative of $y=a^{x}$ would be $y^{\prime}=a^{x} \ln a$.

## Derivatives of Logarithmic Functions

$$
\text { The derivative of } \ln x \text { is } \frac{1}{x} . \quad \text { The derivative of } \log _{a} x \text { is } \frac{1}{x \ln a .} \text {. }
$$

So, how did people figure out how to find the derivatives of logarithmic functions? Well, suppose I really wanted to know the derivative of $\ln x$, at say $x=3$. I could draw a graph, and then sketch the tangent line at $x=3$. The derivative is just the slope of that line, but how to find it?

The answer lies with the inverse function of $\ln x$, which is $e^{x}$. To create the graph of an inverse function, we use the line $y=x$ as a mirror. I can draw the graph of $y=e^{x}$, which is a mirror image of $y=\ln x$, complete with the mirror image of the tangent line:


While I may not know how to find the slope of the original tangent line (shown in blue), I do know how to get the slope of its mirror image. That is just the derivative of $\mathrm{e}^{\mathrm{x}}$, which is also $\mathrm{e}^{\mathrm{x}}$. For the orange tangent line the slope is $\frac{\text { rise }}{\text { run }}=e^{x}$. However, when you take the inverse, $y$ and $x$ are reversed, so that corresponds to $\frac{\text { run }}{\text { rise }}=\frac{1}{\mathrm{e}^{\mathrm{x}}}$ for the blue line. There, it should be easy now. I'll just use $x=3$, and..... Hmm, looking back at the orange line, I see a problem. The slope here isn't really the derivative at $x=3$. It looks more like the derivative at maybe $x=1.1$ or so. Thinking about that for a bit, I realize that while the original point I picked was (3, In 3), the point in the mirror is actually $(\ln 3,3)$. So, I have to take the slope of the orange tangent line at $x=\ln 3$, and then flip it: $\frac{1}{e^{\ln 3}}$. Because $\mathrm{e}^{\mathrm{x}}$ and $\ln \mathrm{x}$ are inverse functions, $\mathrm{e}^{\ln 3}$ is just 3 . The derivative of $y=\ln x$ at $x=3$ is $\frac{1}{3}$.

That works the same way for every value of $x$ : The derivative of $\ln x$ is $\frac{1}{e^{\ln x}}$ :
If $y=\ln x$, then $\frac{d y}{d x}=\frac{1}{x}$


This is a picture of the function $y=\ln (x)$, and what should be its derivative, $y=\frac{1}{x}$. Notice that $\ln (x)$ exists only for positive values of $x$ since negative numbers do not have logarithms. It really looks like we are missing something, so let's change the picture to show $y=\ln |x|$ and its derivative $\mathrm{y}=\frac{1}{\mathrm{x}}$ :


There, that looks a lot better because every value for $x$ except 0 is accounted for. But does $y=\ln |x|$ actually have the same derivative as $y=\ln x$ ? Looking at the graphs, I would say that it is likely. From the left to the center, you can see the natural log function decreasing, slowly at
first and then faster and faster. The supposed derivative is slightly negative at first, indicating a slight decrease, and then becomes more and more negative. To answer the question in a definite way we need to break $y=\ln |x|$ up into its two component parts: $y=\ln x$ for $x>0$, and $y$ $=\ln (-x)$ for $x<0$. We already found the derivative of $y=\ln x$ by using its inverse function. The function $y=\ln (-x)$ has an inverse too. To find that, just switch $x$ and $y$ and solve for $y$ :
$y=\ln (-x)$
$x=\ln (-y)$
$e^{x}=e^{\ln (-y)}$
$e^{x}=-y$
$y=-e^{x} \quad$ This is the inverse function.
To find the derivative of $y=\ln (-x)$, we consider that it is the reciprocal of the derivative of its inverse, taken at the point $\ln (-x)$. The derivative of $y=-e^{x}$ is just $-e^{x}$, so we get $\frac{1}{-e^{\ln (-x)}}=\frac{1}{--x}=$ $\frac{1}{x}$.

Implicit differentiation is actually faster at finding the derivatives of logarithmic functions:
Rewrite $y=\ln x$ as $e^{y}=e^{\ln x}$, which is the same as $e^{y}=x$. Rewrite $y=\ln (-x)$ as $e^{y}=e^{\ln (-x)}$, which turns into $\mathrm{e}^{\mathrm{y}}=-\mathrm{x}$. Now use implicit differentiation:

$$
\begin{array}{ll}
\mathrm{e}^{\mathrm{y}}=\mathrm{x} & \mathrm{e}^{\mathrm{y}}=-\mathrm{x} \\
\mathrm{e}^{\mathrm{y}} \cdot \mathrm{y}^{\prime}=1 & \mathrm{e}^{\mathrm{y}} \cdot \mathrm{y}^{\prime}=-1 \\
\mathrm{y}^{\prime}=\frac{1}{\mathrm{e}^{y}} & \mathrm{y}^{\prime}=\frac{-1}{\mathrm{e}^{y}} \\
\mathrm{y}^{\prime}=\frac{1}{e^{\ln x}} & \mathrm{y}^{\prime}=\frac{-1}{e^{\ln (-x)}} \\
\mathrm{y}^{\prime}=\frac{1}{x} & \mathrm{y}^{\prime}=\frac{1}{x}
\end{array}
$$

Because we know that $\frac{1}{x}$ is the derivative of $y=\ln x$, we can also find the derivatives of logarithmic functions in general. Just like we saw how to express all exponential functions in terms of e, we want to express all logarithmic functions in terms of the natural logarithm, In.

## Example

Find the derivative of $\mathrm{y}=\log _{5} \mathrm{x}$.
To compute a specific value for $y$ when you have a value for $x$ you need to find a better way to express this function, because your calculator probably doesn't do base 5 logarithms. [Note: To follow this example, you need to believe that 5 raised to the power " $\log _{5} x$ " is just $x$. For example, the base $5 \log$ of 125 is 3 , because $5^{3}=125.5$ raised to the power " $\log _{5} 125$ " is 125.]
$y=\log _{5} x$
$5^{y}=5^{\log _{5} x}$
$5^{y}=x$
$\ln \left(5^{y}\right)=\ln x$
$y \ln 5=\ln x$
$y=\frac{1}{\ln 5} \ln x$
Now there is only a natural logarithm in your function, and you can use a calculator to find values, or graph the function. When you go to take the derivative, remember that $\frac{1}{\ln 5}$ is just a constant.
$y^{\prime}=\frac{1}{\ln 5} \cdot \frac{1}{x}=\frac{1}{x \ln 5}$
In general, the derivative of $y=\log _{a} x$ is $y^{\prime}=\frac{1}{x \ln a}$.

Now that you know that the derivative of $\ln \mathrm{x}$ is $\frac{1}{\mathrm{x}}$, you can use implicit differentiation to quickly find the derivative of an exponential function if you can't remember the formula. For example:

$$
y=3^{x}
$$

$\ln y=x \ln 3$
$\frac{1}{y} y^{\prime}=\ln 3$
$y^{\prime}=y \ln 3$ and since $y=3^{x}$ :
$y^{\prime}=3^{x} \ln 3$

## Derivatives of Inverse Functions

The derivative of $f(x)$ at the point $(a, b)$ is $\frac{1}{f^{i n v 1}(b)}$.
$f^{\prime}(x)=\frac{\mathbf{1}}{\mathbf{g}^{\prime}(f(\mathbf{x}))}$, where $g(x)$ is the inverse of $f(x)$.

Continuing from the previous section, we can find the derivative of a function from the derivative of its inverse.

## Example 1

Find the value of the derivative of $f(x)=\sqrt{x}$ at $x=4$ directly, and also from its inverse function.

If you take the derivative of $f(x)=\sqrt{x}$ at $x=4$ directly, you want to re-write the function as $f(x)=x^{1 / 2}$, so $f^{\prime}(x)=\frac{1}{2} x^{-1 / 2}$ which is $\frac{1}{2 \sqrt{x}}$. When $x$ is 4 , that works out to $\frac{1}{4}$.

Now consider the inverse function, which is finv $^{\text {in }}=x^{2}$, with a restricted domain. The derivative of that is 2 x .


The slope of the orange tangent line is $2 x$, where $x$ is 2 rather than 4 . The inverse of that is $\frac{1}{2 x^{\prime}}$ for $x=2$. The slope of the blue tangent line is $\frac{1}{4}$.

## Example 2

Find the value of the derivative of $f(x)$ at the point $(a, b)$ in terms of the derivative of $\mathrm{f}^{\mathrm{fnv}}(\mathrm{x})$.


The tangent line of the inverse function is at the point $(b, a)$, so we want the reciprocal of $\mathrm{f}^{\prime n v}$ (b):
$f^{\prime}(a)=\frac{1}{f^{\text {inv }}(b)}$
Note that $b$ is $f(x)$, so in general:
$f^{\prime}(x)=\frac{1}{f^{\text {inv }}(f(x))}$
or, if $g(x)$ is the inverse of $f(x): f^{\prime}(x)=\frac{1}{g^{\prime}(f(x))}$
[Of course it also works the other way around: $\mathrm{f}^{\text {inv }}(\mathrm{x})=\frac{1}{\mathrm{f}^{\prime}\left(\mathrm{f}^{\operatorname{inv}}(\mathrm{x})\right)}$ or $\mathrm{g}^{\prime}(\mathrm{x})=\frac{1}{\mathrm{f}^{\prime}(\mathrm{g}(\mathrm{x}))}$ ]
This formula is useful when the derivative of a function is hard to find or totally unknown.

Now we can verify the general formula for our specific example of $f(x)=\sqrt{x}$ and $f^{\text {inv }}(x)=x^{2}$ :
$f^{\prime}(x)=\frac{1}{f^{i n v}(f(x))}$
$\mathrm{f}^{\prime}(\mathrm{x})=\frac{1}{2(\mathrm{f}(\mathrm{x}))}=\frac{1}{2 \sqrt{\mathrm{x}}}$
Although you wouldn't have a particular reason to do this the other way around, you could actually find derivative of $f(x)=x^{2}$ by using the derivative of the inverse function $f^{\text {inv }}(x)=\sqrt{x}$ :
$f^{\prime}(x)=\frac{1}{f^{\text {inv }}(f(x))}=\frac{1}{\frac{1}{2 \sqrt{f(x)}}}=\frac{1}{\frac{1}{2 \sqrt{x^{2}}}}$
Because $1 \div \frac{1}{2 \sqrt{\mathrm{x}^{2}}}=1 \cdot \frac{2 \sqrt{\mathrm{x}^{2}}}{1}$, the answer is $2|\mathrm{x}|$.
The absolute value sign appears here because we are considering $f(x)=x^{2}$ specifically as the inverse of $g(x)=\sqrt{x}$, so we only get the part of the function where $x$ is positive.

Here is recap using implicit differentiation, which is more efficient:
The inverse of $y=x^{2}$ is $x=y^{2}$ :
$y^{2}=x$
$2 y \cdot y^{\prime}=1$
$y^{\prime}=\frac{1}{2 y}$ and $y=\sqrt{x}$, so
$y^{\prime}=\frac{1}{2 \sqrt{x}}$

## Inverse Trigonometric Functions

| Arcsine | Domain | Range |  |
| :---: | :---: | :---: | :---: |
|  | -1 to 1 | $-\pi / 2$ to $\pi / 2$ radians |  |
| Arccosecant | $(-\infty,-1] \cup[1, \infty)$ | $-\pi / 2$ to $\pi / 2$ radians, not including 0 |  |
| Arccosine | -1 to 1 | 0 to $\pi$ radians ( $0^{\circ}$ to $180^{\circ}$ ) |  |
| Arcsecant | $(-\infty,-1] \cup[1, \infty)$ | 0 to $\pi$ radians, not including $\pi / 2$ |  |
| Arctangent | any real number | $-\pi / 2$ to $\pi / 2$ radians (not including endpoints) |  |
| Arccotangent | any real number | 0 to $\pi$ (not including endpoints) |  |
|  |  | OR $-\pi / 2$ to $\pi / 2$, not including 0 |  |
| Function | Derivative | Function | Derivative |
| $f(x)=\sin ^{-1} x$ | $\frac{1}{\sqrt{1-x^{2}}}$ | $f(x)=\sec ^{-1} x$ | $\frac{1}{\|x\| \sqrt{x^{2}-1}}$ |
| $f(x)=\cos ^{-1} x$ | $\frac{-1}{\sqrt{1-x^{2}}}$ | $f(x)=\csc ^{-1} x$ | $\frac{-1}{\|x\| \sqrt{x^{2}-1}}$ |
| $f(x)=\tan ^{-1} x$ | $\frac{1}{1+x^{2}}$ | $f(x)=\cot ^{-1} x$ | $\frac{-1}{1+\mathrm{x}^{2}}$ |

By now it will be easy for you to find the derivative of $y=\sin x$, but what is the derivative of the inverse sine function? At this point you may not quite remember what an inverse sine function even is, so let's do a quick review.

If we have an angle of $\pi / 6$ radians ( 30 degrees), we can figure out that the sine $1 / 2$. The inverse sine function takes the value of the sine, and gives us the angle. The inverse sine function is called the arcsine, and it is often written, confusingly, as $\sin ^{-1} x$. This does not mean $\frac{1}{\sin x^{\prime}}$, which is the cosecant. If we know that the sine of an angle is $1 / 2$, we might guess that the
angle is $\pi / 6$ radians. However, it could also be $5 \pi / 6$ radians $\left(150^{\circ}\right)$, or $21 / 6 \pi$ radians, or.... The possibilities are endless because the sine function repeats endlessly. An inverse sine function can only be created by taking a portion of the sine curve that is small enough that it does not return multiple angles for a given value of $\sin (x)$. The part that is used for this purpose is the portion of the sine curve between $-\pi / 2$ and $\pi / 2$ radians ( $-90^{\circ}$ to $90^{\circ}$ ). This may seem a bit hard to remember, but just think about what part of the unit circle you would choose if you were in charge. You would have to allow for the full range of values of the sine, which is from -1 to 1 . You could start that at $\pi / 2$ and go to $\frac{3 \pi}{2}$ radians, but chances are that you'd leave it right where it is now just to avoid all those awkward fractions in between. As a result, the inverse sine function, also known as the arcsine function, will always return angles between $-\pi / 2$ and $\pi / 2$ radians. Any sine value is a valid input for this function. Recall that the sine is always between -1 and 1, so to avoid getting a reproachful error message from your calculator don't try to stick inappropriate values into the arcsine function.

In the same way, mathematicians have defined an inverse cosine function, which returns angles between 0 and $\pi$ radians. Again the input had to be between -1 and 1 , so the most convenient interval to choose was 0 to $\pi$ radians. This allows for the full range of values of the cosine, and returns one unique angle for each one.

If you look at the tangent function, you can see that over the interval $-\pi / 2$ to $\pi / 2$ radians the tangent goes from its minimum value (-infinity) to its maximum value (+infinity). Here again there are other intervals you could choose, but $-\pi / 2$ to $\pi / 2$ is the most convenient one. The arctangent function gives you angles between $-\pi / 2$ and $\pi / 2$ radians for any tangent value you enter.

|  | Domain | Range |
| :--- | :--- | :--- |
| Arcsine | -1 to 1 | $-\pi / 2$ to $\pi / 2$ radians |
| Arccosecant | $(-\infty,-1] \cup[1, \infty)$ | $-\pi / 2$ to $\pi / 2$ radians, not including 0 |
| Arccosine | -1 to 1 | 0 to $\pi$ radians $\left(0^{\circ}\right.$ to $\left.180^{\circ}\right)$ |
| Arcsecant | $(-\infty,-1] \cup[1, \infty)$ | 0 to $\pi$ radians, not including $\pi / 2$ |
| Arctangent | any real number | $-\pi / 2$ to $\pi / 2$ radians (not including endpoints) |
| Arccotangent | any real number | 0 to $\pi$ (not including endpoints) |

The range of the inverse cotangent function can also be $-\pi / 2$ to $\pi / 2$ radians (not including endpoints) just like the inverse tangent, but many people prefer 0 to $\pi$ so that the range is continuous.

As with any pair of inverse functions, $\sin \left(\sin ^{-1} x\right)=x$ and $\sin ^{-1}(\sin x)=x$. Be careful though, as $\sin \left(\sin ^{-1} 2\right)$ is not equal to 2 because of the restricted domain of the arcsine function.

We can find the derivative of the inverse sine function using the methods we just learned for finding derivatives of inverse functions.

Consider $\mathrm{y}=\sin \mathrm{x}$ and $\mathrm{y}_{\mathrm{inv}}=\sin ^{-1} \mathrm{x}$. To take the derivative of $\mathrm{y}_{\mathrm{inv}}$, we would use the general formula $f^{\prime}(x)=\frac{1}{g^{\prime}(f(x))^{\prime}}$, which in this case looks like $y^{\prime}$ inv $=\frac{1}{y^{\prime}\left(y_{\text {inv }}\right)}$.
$y_{\text {inv }}^{\prime}=\frac{1}{\cos y_{\text {inv }}}=\frac{1}{\cos \left(\sin ^{-1} x\right)}$
Hmm, that is not so convenient. First we have to find the angle that goes with x , and then take the cosine of it. Fortunately we can do this rather easily by drawing a triangle to represent the situation. If we want the inverse sine of $x$, that means we are looking for some angle that has a sine of $x$. The simplest way to draw that would be to give the triangle a hypotenuse of 1 and an opposite side of $x$ :


By looking at the picture carefully you can see that the cosine of that same angle is $\sqrt{1-\mathrm{x}^{2}}$.
The disadvantage of such pictures is that they do not show potential negative square roots. In this particular case that doesn't matter, because the cosine is positive for the entire range, or output, of the inverse sine function. Do not forget to check for negatives if you are using a picture!

We can do the same thing without drawing a picture because we know the relationship between the sine and the cosine: $\sin ^{2} x+\cos ^{2} x=1$. Simple algebra tells us that $\cos x=$ $\pm \sqrt{1-\sin ^{2} x}$. Here the cosine will be positive so we can say that $\cos x=\sqrt{1-\sin ^{2} x}$. So, instead of taking the cosine of $\left(\sin ^{-1} x\right)$ we can take $\sqrt{1-\sin ^{2}\left(\sin ^{-1} x\right)}$. This really means
$\sqrt{1-\left(\sin \left(\sin ^{-1} x\right)\right)^{2}}$. If you feel comfortable with the idea that $\sin \left(\sin ^{-1} x\right)=x$, you can easily convert this to $\sqrt{1-x^{2}}$. If you don't feel comfortable, try it out with some real angle values until you do. The result of this substitution is:
$y^{\prime}{ }_{\text {inv }}=\frac{1}{\cos y_{\text {inv }}}=\frac{1}{\cos \left(\sin ^{-1} x\right)}=\frac{1}{\sqrt{1-x^{2}}}$
You probably can't wait to see if you get the same result using implicit differentiation, so let's try that:
$y=\sin ^{-1} x$
$x=\sin y$
$1=\cos y y^{\prime}$
$y^{\prime}=\frac{1}{\cos y}=\frac{1}{\cos \left(\sin ^{-1} x\right)}=\frac{1}{\sqrt{1-x^{2}}}$
You can use these same methods to find the derivative of the inverse cosine function. Try it out for yourself. Use the fact that the sine is never negative over the range of the inverse cosine function.

With a little imagination you can also find the derivative of the inverse tangent function. Just divide both sides of $\sin ^{2} x+\cos ^{2} x=1$ by $\cos ^{2} x$ to change this identity to $\tan ^{2} x+1=\sec ^{2} x$.

Function Derivative
$f(x)=\sin ^{-1} x \quad \frac{1}{\sqrt{1-x^{2}}}$
$f(x)=\cos ^{-1} x \quad \frac{-1}{\sqrt{1-x^{2}}}$
$f(x)=\tan ^{-1} x \quad \frac{1}{1+x^{2}}$

## Example

Find the derivative of $f(x)=\tan (\arcsin x)$.

Arcsin $x$ is the same as $\sin ^{-1} x$. First take the derivative of the function as it is written:
$f^{\prime}(x)=\sec ^{2}(\arcsin x) \cdot \frac{1}{\sqrt{1-x^{2}}} \cdot \sec ^{2}(\arcsin x)$ means $(\sec (\arcsin x))^{2}$, so we can simplify $\sec \left(\sin ^{-1} x\right)$ and then square it. Look at the picture above to see that $\sec \left(\sin ^{-1} x\right)=\frac{1}{\sqrt{1-x^{2}}}$. Now square that to get $\frac{1}{1-\mathrm{x}^{2}}$, so the final answer is $\frac{1}{1-\mathrm{x}^{2}} \cdot \frac{1}{\sqrt{1-\mathrm{x}^{2}}}$ which is the same as $\left(1-\mathrm{x}^{2}\right)^{-3 / 2}$.

## Example

Find the derivative of the inverse trigonometric function $f(x)=\operatorname{arcsec} x$.
If $y=\sec ^{-1} x$, then $x=\sec y$. Differentiate implicitly:
$1=(\sec y)(\tan y)\left(y^{\prime}\right)$
$1=\sec \left(\sec ^{-1} x\right) \tan \left(\sec ^{-1} x\right) y^{\prime}$
$y^{\prime}=\frac{1}{x \cdot \tan \left(\sec ^{-1} x\right)}$
From the identity $\tan ^{2} x+1=\sec ^{2} x$, we get that $\tan x= \pm \sqrt{\sec ^{2} x-1}$. Unfortunately the tangent can actually be negative over the range of the inverse secant function, which runs from 0 to $\pi$, not including $\frac{\pi}{2}$.
$y^{\prime}=\frac{1}{x \cdot\left( \pm \sqrt{\left.\sec ^{2}\left(\sec ^{-1} x\right)-1\right)}\right.}$
That doesn't look nice at all, but we can fix it by realizing that the tangent will be negative only for negative values of $x$. When $x$ is negative, we'll get $y^{\prime}=\frac{1}{x \cdot\left(-\sqrt{\sec ^{2}\left(\sec ^{-1} x\right)-1}\right)}$, which is positive because we are multiplying by a negative x . When x is positive, $\mathrm{y}^{\prime}=$
$\frac{1}{\mathrm{x} \cdot\left(+\sqrt{\left.\sec ^{2}\left(\sec ^{-1} \mathrm{x}\right)-1\right)}\right.}$, which is also positive. This means that we can just indicate that the expression will be positive by using the absolute value of x :
$y^{\prime}=\frac{1}{|x| \sqrt{x^{2}-1}}$.
When you look at the graph of the inverse secant function, you can see that it is increasing over its entire range, so the derivative must be positive. Just like the function, the derivative doesn't exist at $\mathrm{x}=0$.

In the same way, you can show that the derivative of the inverse cosecant function is $\frac{-1}{|\mathrm{x}| \sqrt{\mathrm{x}^{2}-1}}$.

## Example

Find the derivative of the inverse trigonometric function $f(x)=\operatorname{arccot} x$.
$y=\cot ^{-1}(x)$, so $x=\cot y$. Again implicit differentiation provides the fastest result:
$1=-\csc ^{2}(y) y^{\prime}$
$y^{\prime}=\frac{-1}{\csc ^{2} y}$
$y^{\prime}=\frac{-1}{\csc ^{2}\left(\cot ^{-1}(x)\right)}$
Since $\csc ^{2} x=\cot ^{2} x+1$ :
$y^{\prime}=\frac{-1}{\cot ^{2}\left(\cot ^{-1}(x)\right)+1}$
That may look a bit confusing, but just rewrite it to show what it actually means:
$y^{\prime}=\frac{-1}{\left(\cot \left(\cot ^{-1}(x)\right)\right)^{2}+1}$
$y^{\prime}=\frac{-1}{x^{2}+1}$
There is no concern here about whether the cotangent is positive or negative, since we didn't have to take a square root.

## VII. The Integral - An Amazing Discovery

## Riemann Sums

Riemann Sums use rectangles or trapezoids to estimate the area under a curve.

Rectangles: Use $f(x)$ at the start, middle or end of each interval as required.
Trapezoids: Use the average value of $f(x)$ over the interval.

The real area is $\lim _{\mathrm{n} \rightarrow \infty} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{f}\left(\mathrm{x}_{\mathrm{i}}\right) \Delta \mathrm{x}=\lim _{\mathrm{n} \rightarrow \infty} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{f}(\mathrm{a}+\mathrm{i} \Delta \mathrm{x}) \Delta \mathrm{x}$, where $\Delta \mathrm{x}=\frac{\mathrm{b}-\mathrm{a}}{\mathrm{n}}$.
Use these summation formulas: $\quad \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{ci}=\mathrm{c} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{i}$

$$
\begin{aligned}
& \sum_{i=1}^{\mathrm{n}} \mathrm{i}=\frac{\mathrm{n}}{2}(1+\mathrm{n}) \\
& \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{i}^{2}=\frac{\mathrm{n}(\mathrm{n}+1)(2 \mathrm{n}+1)}{6}
\end{aligned}
$$

The derivative makes a lot of practical sense when you think of it as speed, or more precisely, velocity (speed with direction). To find the velocity you start with a distance vs. time graph, and take the derivative of the function that describes the motion. Now consider the reverse: start with velocity and find the distance traveled. We know that velocity = distance/time, so that means distance = velocity $\cdot$ time. When the velocity is constant, the distance traveled is easy to find. We simply multiply the speed by the elapsed time. For example, 40 miles per hour times 10 hours $=400$ miles. Notice that this corresponds to the area under the velocity line:


If we want the distance traveled between two specific time points $t_{1}$ and $t_{2}$, we can multiply the velocity by $\mathrm{t}_{2}-\mathrm{t}_{1}$. The distance traveled between $\mathrm{t}=8$ hours and $\mathrm{t}=10$ hours is $40 \mathrm{mph} \cdot 2 \mathrm{hrs}$ $=80$ miles. For this graph such calculations are very easy.

Take a look at the following velocity graph. The speed is increasing at a steady rate.


To get the distance traveled, we need to take the average speed over a certain time interval, and multiply it by the elapsed time. Between $t=0$ and $t=2$ hours, the speed changes from 0 mph to 40 mph . The average speed over this time interval is 20 miles per hour, so the distance
traveled is 20 miles per hour $\cdot 2$ hours $=40$ miles. Again notice that this corresponds to the area under the velocity line from $t=0$ to $t=2$.

Next, consider this velocity graph:


How can we find the distance traveled now? One way to do that is to split up the total time interval between 0 and 2 hours into much smaller intervals. Then we could estimate the average velocity during one small time interval, and multiply it by the time elapsed over that interval. Then we go to the next interval and do the same, and so on until we are done, and then add everything together. Does that sound tedious? Welcome to Riemann Sums! The only consolation I can offer you here is that this is just one brief section of your course.

The area under a curve can represent far more than just "distance traveled". Anytime that a quantity can be calculated by multiplying two different variables, you can look at it as the area under a line or a curve. For example, in physics, Work is Force times distance. When the force is constant the work done is easy to calculate, but in some cases the force changes. If it changes at a steady rate you'll get a graph with a straight line and you can use the average force to determine the work done. However, if the force changes at a changing rate your graph will be a curve, and the area is not so easy to find. There are many practical uses for areas under curves, but a very important one is in determining the proper dosages of medicines. When you take a pill, not all of the medicine is absorbed, and your body begins to eliminate what does get absorbed. The total effective amount of that drug is determined by its concentration in your blood over time. When researchers graph that the result is usually a curve, so they must find the area underneath that curve to determine the right dosage.

When you use Riemann Sums, you will be dividing the area under a curve into a limited number of sections, called intervals. Sometimes you will use the function value at the start of each interval to do the estimate. This is often called using left endpoints. You may also be asked to use the function value at the end of each interval, in which case you are using right endpoints. Right endpoints are shown below. Notice that the area will be overestimated because the function is increasing.


At other times you will use the x-coordinate in the middle of the interval to estimate the value of the function for the entire interval. This is called using midpoints. You may think that midpoints provide a more accurate estimate of the area under the curve. This is in fact true when you are using a limited number of intervals.

Rather than creating rectangles to approximate the area under a curve, we can use trapezoids. That creates a nicer fit. As you may recall from geometry, the area of a trapezoid is found by taking the average of the two bases and multiplying that by the height. Because the trapezoids here are thin and the bases form the sides, you may not initially recognize them as trapezoids. The picture below shows just four intervals using the trapezoidal method, because I don't have the patience to draw more. All you need for your calculation is the function value at the start of the interval, and at the end. Take the average, and multiply it by the width of the interval, which is actually the height of the trapezoid.


If you don't like the calculations that come with this method, you can find the Riemann Sum using right endpoints and the Sum using left endpoints, and then take the average of the two Sums. That gives the same result.

## Example

Estimate the area under the curve $y=10 x^{2}$, between $x=0$ and $x=2$, using the right endpoints (the ending values) of 4 separate intervals.

You may notice that this is the actual curve shown in the previous illustration. To do this rightendpoint estimate, we pretend that the $y$-value over the entire interval is the same as it is at the right endpoint. The total $x$-distance is 2 , and we need to divide that into 4 equal sections, creating 4 rectangles like this:


The length of each section is 2 divided by 4 or $\frac{1}{2}$. The first interval runs from $x=0$ to $x=\frac{1}{2}$. At the end of this interval, $y=10 \cdot\left(\frac{1}{2}\right)^{2}=\frac{10}{4}$. At the end of the second interval, $y=10 \cdot 1^{2}$. The end of the third interval has a $y$-value of $10 \cdot\left(\frac{3}{2}\right)^{2}=\frac{90}{4}$, and at the end of the fourth interval we have a $y$-value of 40 . The area would then be composed of 4 rectangles with the following areas:
$\frac{1}{2} \cdot \frac{10}{4}=\frac{10}{8}$
$\frac{1}{2} \cdot 10=5$
$\frac{1}{2} \cdot \frac{90}{4}=\frac{90}{8}$
$\frac{1}{2} \cdot 40=20$
The total area is $25+\frac{100}{8}=37.5$.

Your calculus course will take the trouble to impress upon you that the more intervals you use, the more accurate your estimate becomes. From here, it is not such a big leap to think that we could get a really accurate value for the area under the curve if we used infinitely many, infinitely small intervals. When you do that, it no longer matters whether you use right endpoints, left endpoints, or midpoints, since each interval is infinitely small.

## Example

Find an expression for the area under the curve $y=10 x^{2}$, between $x=0$ and $x=2$, using the right endpoints of $n$ separate intervals.

This problem is really the same as the previous one; it is just more general. The total x-distance is still 2 , and we need to divide that into $n$ equal sections. Therefore the width of each section is 2 divided by $n$ or $\frac{2}{\mathrm{n}}$. The first interval runs from $\mathrm{x}=0$ to $\mathrm{x}=\frac{2}{\mathrm{n}}$. At the end of this interval, $\mathrm{y}=$ $10 \cdot\left(\frac{2}{n}\right)^{2}$. That is straightforward, but where is the end of the second interval? If you think about it for a bit you will see that it is easiest to figure things out like this:
end of first interval: $\frac{2}{\mathrm{n}}$
end of second interval: $\frac{2}{n}+\frac{2}{n}=\frac{2 \cdot 2}{n}$
end of third interval: $\frac{2}{\mathrm{n}}+\frac{2}{\mathrm{n}}+\frac{2}{\mathrm{n}}=\frac{2 \cdot 3}{\mathrm{n}}$
end of fourth interval: $\frac{2}{\mathrm{n}}+\frac{2}{\mathrm{n}}+\frac{2}{\mathrm{n}}+\frac{2}{\mathrm{n}}=\frac{2 \cdot 4}{\mathrm{n}}$
Because this goes up in a nice orderly way, we can use a counter variable. The letter n has already been used in this problem, but we can turn to something from computer science. The increment counter i is commonly used in iterations, where the program instructs the computer to repeatedly perform a task (and increase the value of $i$ by 1 ) until the counter i reaches a preset value. Because $i$ is already in use for the imaginary numbers, you may instead see $j, k$ or some other variable. Caution: when using i as a counter, do not square it to get -1 !

End of $\mathrm{i}^{\text {th }}$ interval: $\frac{2 \cdot \mathrm{i}}{\mathrm{n}}$
We calculate the area of each rectangle, until the counter i reaches n (there are n intervals, so n rectangles). Each time, the area of the rectangle is the width times the height, or $\frac{2}{\mathrm{n}}$ times $10 \cdot\left(\frac{2 i}{n}\right)^{2}$. The first value of $i$ is 1 , so the area of the first rectangle is $\frac{80}{n^{3}}$. Once we are done, or even while we are working, we add all of the areas together. There is a nice shorthand notation for this. We need "the sum of" $\frac{2}{n} \cdot 10\left(\frac{2 \mathrm{i}}{\mathrm{n}}\right)^{2}$, for $\mathrm{i}=1$ to $\mathrm{i}=\mathrm{n}$. The sum can be indicated by the Greek letter $S, \Sigma$, which unfortunately is a bit scary-looking (no offense to the Greeks). The final notation is $\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{2}{\mathrm{n}} \cdot 10\left(\frac{2 \mathrm{i}}{\mathrm{n}}\right)^{2}$, which then simplifies to $\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{80 \mathrm{i}^{2}}{\mathrm{n}^{3}}$. Notice that the counter values are indicated from $\mathrm{i}=1$ at the bottom to simply n at the top. Let's check out this expression for 4 intervals, because that is what we used before. $n$ is a variable which will have a value of 4 in this case, but $i$ is a counter that changes with each step. For the first interval, $i=1$, for the second interval $i=2$, and so on:
$\frac{80 \cdot 1}{4^{3}}+\frac{80 \cdot 4}{4^{3}}+\frac{80 \cdot 9}{4^{3}}+\frac{80 \cdot 16}{4^{3}}=\frac{10}{8}+\frac{40}{8}+\frac{90}{8}+\frac{160}{8}=\frac{300}{8}=37.5$

To apply the idea of limits to integrals, we will use n intervals rather than some definite number like 4 or 10. Now the area under the curve becomes the limit of the Riemann Sums as $n$ goes to
infinity. The idea is simple, but just like with a limited number of intervals the details are a bit tedious to work out.

To get the area under the curve from $x=a$ to $x=b$, we will divide up the distance $b-a$ into $n$ intervals that are $\Delta x$ wide. At the end of each interval (right endpoints), we will evaluate $f(x)$. All of these different $x$ values for which we need to find $f(x)$ will be distinguished by a subscript: $x_{1}, x_{2}, x_{3}$, etc., or in general, $x_{i}$. So, for the general interval where the ending $x$ value is $x_{i}$, the area of that section is $f\left(x_{i}\right)$, the height, times the width, $\Delta x$. To get the total, we sum up all of the separate areas:
$\mathrm{f}\left(\mathrm{x}_{1}\right) \Delta \mathrm{x}+\mathrm{f}\left(\mathrm{x}_{2}\right) \Delta \mathrm{x}+\mathrm{f}\left(\mathrm{x}_{3}\right) \Delta \mathrm{x}+\mathrm{f}\left(\mathrm{x}_{4}\right) \Delta \mathrm{x}+\mathrm{f}\left(\mathrm{x}_{5}\right) \Delta \mathrm{x}+\ldots+\mathrm{f}\left(\mathrm{x}_{\mathrm{i}}\right) \Delta \mathrm{x}$
There are a total of $n$ terms here, because we are using $n$ intervals. As a shortcut, we use summation notation to write this:
$\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{f}\left(\mathrm{x}_{\mathrm{i}}\right) \Delta \mathrm{x}=\mathrm{f}\left(\mathrm{x}_{1}\right) \Delta \mathrm{x}+\mathrm{f}\left(\mathrm{x}_{2}\right) \Delta \mathrm{x}+\mathrm{f}\left(\mathrm{x}_{3}\right) \Delta \mathrm{x}+\mathrm{f}\left(\mathrm{x}_{4}\right) \Delta \mathrm{x}+\mathrm{f}\left(\mathrm{x}_{5}\right) \Delta \mathrm{x}+\ldots+\mathrm{f}\left(\mathrm{x}_{\mathrm{i}}\right) \Delta \mathrm{x}$
Here the summation notation uses a counter, i , with the count starting at $\mathrm{i}=1$ and ending when $i$ is equal to $n$. Now we can say that the area $A$ is equal to the limit, as $n$ goes to infinity, of $\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{f}\left(\mathrm{x}_{\mathrm{i}}\right) \Delta \mathrm{x}$, which is the important part. If I had a choice, I would want to stop right here and continue on to integrals. If you have a choice, I suggest you do the same.

To actually do the calculations, you need to keep in mind that $\Delta x$ is equal to $\frac{b-a}{n}$, because that is the width of each interval. The end of the first interval occurs at an $x$-value of $a+\Delta x$, the end of the second interval is at $a+2 \Delta x$, the end of the third is at $a+3 \Delta x$, and so on. The end of each interval in general is at $a+i \Delta x$. If we determine the function value at the end of each interval, that would be $f(a+i \Delta x)$. So, we have to find the sum, $\sum_{i=1}^{n} f\left(x_{i}\right) \Delta x$, which is $\sum_{i=1}^{n} f(a+i \Delta x) \Delta x$, or $\sum_{i=1}^{n} f\left(a+i \frac{b-a}{n}\right) \frac{b-a}{n}$.

To use left endpoints, we need to measure $f(x)$ at the beginning of each interval. In that case, we want to start at a rather than at $a+i \Delta x$. This can be accomplished by setting a starting value of 0 for $\mathrm{i}: \sum_{\mathrm{i}=0}^{\mathrm{n}-1} \mathrm{f}(\mathrm{a}+\mathrm{i} \Delta \mathrm{x}) \Delta \mathrm{x}$. Notice that the ending value is now $\mathrm{n}-1$.

## Example

Find the area under the graph of $y=x^{2}$, from $x=1$ to $x=2$, using right endpoints.

The area is the limit, as $n$ goes to infinity, of $\sum_{i=1}^{n} f\left(x_{i}\right) \Delta x$. Using right endpoints, we can write that as $\lim _{\mathrm{n} \rightarrow \infty} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{f}(\mathrm{a}+\mathrm{i} \Delta \mathrm{x}) \Delta \mathrm{x}$ or $\lim _{\mathrm{n} \rightarrow \infty} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{f}(1+\mathrm{i} \Delta \mathrm{x}) \Delta \mathrm{x}$, since $\mathrm{x}=1$ is our starting point for the interval from $a$ to $b$.
$\Delta x$ is $\frac{b-a}{n}$, or in this case $\frac{2-1}{n}$. Stick that into the sum, to get $\sum_{i=1}^{n} f\left(1+i \frac{1}{n}\right) \frac{1}{n}$. Because we know that $f(x)$ is $x^{2}$, we write that as $\sum_{i=1}^{n}\left(1+i \frac{1}{n}\right)^{2} \frac{1}{n^{\prime}}$ which is just shorthand for $\left(1+\frac{1}{n}\right)^{2} \frac{1}{n}+\left(1+2 \cdot \frac{1}{n}\right)^{2} \frac{1}{n}+\left(1+3 \cdot \frac{1}{n}\right)^{2} \frac{1}{n}+\ldots+\left(1+n \cdot \frac{1}{n}\right)^{2} \frac{1}{n}$.

To calculate this sum, $\sum_{\mathrm{i}=1}^{\mathrm{n}}\left(1+\mathrm{i} \frac{1}{\mathrm{n}}\right)^{2} \frac{1}{\mathrm{n}}$, we should first simplify $\left(1+\mathrm{i} \cdot \frac{1}{\mathrm{n}}\right)^{2} \frac{1}{\mathrm{n}}$ : $\frac{1}{n}\left(1+\frac{i}{n}\right)^{2}=\frac{1}{n}\left(1+\frac{i}{n}\right)\left(1+\frac{i}{n}\right)=\frac{1}{n}\left(1+\frac{2 i}{n}+\frac{\mathrm{i}^{2}}{\mathrm{n}^{2}}\right)=\frac{1}{\mathrm{n}}+\frac{2 \mathrm{i}}{\mathrm{n}^{2}}+\frac{\mathrm{i}^{2}}{\mathrm{n}^{3}}$

To actually be able to find the sum $\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{1}{\mathrm{n}}+\frac{2 \mathrm{i}}{\mathrm{n}^{2}}+\frac{\mathrm{i}^{2}}{\mathrm{n}^{3}}$, you need to know a few things about sums. The first thing to realize is that sums with a " + " sign can be split, because the order in which you add the terms doesn't really matter:

$$
\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{1}{\mathrm{n}}+\frac{2 \mathrm{i}}{\mathrm{n}^{2}}+\frac{\mathrm{i}^{2}}{\mathrm{n}^{3}}=\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{1}{\mathrm{n}}+\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{2 \mathrm{i}}{\mathrm{n}^{2}}+\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{\mathrm{i}^{2}}{\mathrm{n}^{3}}
$$

In general, $\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{a}+\mathrm{b}=\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{a}+\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{b}$
Next, if there is only a constant in your sum, you will be adding up that constant over and over till you get to the end. $\sum_{i=1}^{n} 4=4+4+4+\ldots$. , which is $n \cdot 4$ or $4 n$.
$\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{c}=\mathrm{n} \cdot \mathrm{c}$
$\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{1}{\mathrm{n}}$ just means keep adding $\frac{1}{\mathrm{n}}$, since there is no i . When you are finished with that you will have added up $n$ times $\frac{1}{n}$ for a total sum of $\frac{n}{n}$, which is 1 .

That helps you understand what is going on, but is often easiest to factor things out of the sum to make it simpler.
$5+5 x+5 x^{2}$, can be written as $5\left(1+x+x^{2}\right)$, and it doesn't matter if there are $n$ terms instead of three.

You can also find $\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{1}{\mathrm{n}}$ by factoring out $\frac{1}{\mathrm{n}}$, because here n represents a number that will remain constant for each iteration (each increase of the counter).
$\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{1}{\mathrm{n}}=\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{1}{\mathrm{n}} \cdot 1=\frac{1}{\mathrm{n}} \sum_{\mathrm{i}=1}^{\mathrm{n}} 1$
$\frac{1}{\mathrm{n}} \sum_{\mathrm{i}=1}^{\mathrm{n}} 1=\frac{1}{\mathrm{n}} \cdot \mathrm{n}=1$.
[As you are looking at all of this it may seem to make sense, but when you are working by yourself you may notice that you can factor out $\frac{1}{\mathrm{n}}$, and then let n go to infinity. Won't that cause a multiplication by zero? Well, $\frac{1}{\mathrm{n}}$ is not actually zero because we are talking about a limit, and the associated sum is also a limit. For $\frac{1}{\mathrm{n}} \sum_{\mathrm{i}=1}^{\mathrm{n}} 1, \frac{1}{\mathrm{n}}$ is heading to zero as n gets ever larger, while $\sum_{\mathrm{i}=1}^{\mathrm{n}} 1$ is going to infinity. The limit of the whole thing is 1 .]

The same goes for $\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{2 \mathrm{i}}{\mathrm{n}^{2}}$, which really means $\frac{2}{\mathrm{n}^{2}} \cdot 1+\frac{2}{\mathrm{n}^{2}} \cdot 2+\frac{2}{\mathrm{n}^{2}} \cdot 3+\ldots+\frac{2}{\mathrm{n}^{2}} \cdot \mathrm{n}$ or $\frac{2}{n^{2}}(1+2+3+\ldots+n)$

So, for $\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{2 \mathrm{i}}{\mathrm{n}^{2}}$ you can factor out $\frac{2}{\mathrm{n}^{2}}$. The counter, i , cannot be factored out because it has a different value for each term.
$\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{ci}=\mathrm{c} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{i}$
That leaves you with $\frac{2}{\mathrm{n}^{2}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{i}$.
$\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{i}$ means start with $\mathrm{i}=1$, and keep adding until $\mathrm{i}=\mathrm{n}: 1+2+3+4+\ldots+\mathrm{n}$. The method for adding up a series like this was discovered by a young student who was told to add up the first 100 counting numbers. He could have done that the hard way, but he was smart and realized that these numbers come in pairs. The first number plus the last number add to 101. The second number, 2 , and the second-last number, 99 , also add to 101 . This holds all the way, so that the two middle numbers, 50 and 51 again add up to 101 . All we really have to do is say that there are 50 pairs of numbers, each with a sum of 101 . The sum is 5050 . That sure beats adding all those numbers one by one! The formula for using this trick with $n$ numbers would be $\frac{\mathrm{n}}{2}(1+\mathrm{n})$ :
$\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{i}=\frac{\mathrm{n}}{2}(1+\mathrm{n})$
So, $\frac{2}{\mathrm{n}^{2}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{i}=\frac{2}{\mathrm{n}^{2}} \cdot \frac{\mathrm{n}}{2}(1+\mathrm{n})$. That last part can be multiplied out:
$\frac{2}{\mathrm{n}^{2}} \cdot \frac{\mathrm{n}}{2}(1+\mathrm{n})=\frac{2 \mathrm{n}}{2 \mathrm{n}^{2}}+\frac{2 \mathrm{n}^{2}}{2 \mathrm{n}^{2}}=\frac{1}{\mathrm{n}}+1$

That takes care of the first two sums, $\sum_{i=1}^{n} \frac{1}{n}$ and $\sum_{i=1}^{n} \frac{2 \mathrm{i}}{\mathrm{n}^{2}}$. So far we have $1+\frac{1}{\mathrm{n}}+1$. Now for the last sum, $\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{\mathrm{i}^{2}}{\mathrm{n}^{3}}$. Again you can see that you may factor out $\frac{1}{\mathrm{n}^{3}}$, to get $\frac{1}{\mathrm{n}^{3}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{i}^{2}$. If you let $i$ increase by one each time, and square it, you get the series $1+4+9+16+\ldots+n^{2}$.
Fortunately there is a formula for the sum of the first $n$ squares, which is $\frac{n(n+1)(2 n+1)}{6}$ :
$\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{i}^{2}=\frac{\mathrm{n}(\mathrm{n}+1)(2 \mathrm{n}+1)}{6}$
By using this formula we can say that $\frac{1}{\mathrm{n}^{3}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{i}^{2}=\frac{1}{\mathrm{n}^{3}} \cdot \frac{\mathrm{n}(\mathrm{n}+1)(2 \mathrm{n}+1)}{6}$. If you multiply that out you get $\frac{2 \mathrm{n}^{3}+3 \mathrm{n}^{2}+1}{6 \mathrm{n}^{3}}$, which can be split up into separate fractions and simplified: $\frac{1}{3}+\frac{1}{2 \mathrm{n}}+\frac{1}{6 \mathrm{n}^{3}}$. There, the total sum $\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{1}{\mathrm{n}}+\frac{2 \mathrm{i}}{\mathrm{n}^{2}}+\frac{\mathrm{i}^{2}}{\mathrm{n}^{3}}$ is equal to $1+\frac{1}{\mathrm{n}}+1+\frac{1}{3}+\frac{1}{2 \mathrm{n}}+\frac{1}{6 \mathrm{n}^{3}}$. If you can still remember what you were trying to do when you started this problem, you can now take the limit of that as $n$ goes to infinity:
$\lim _{\mathrm{n} \rightarrow \infty} \sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{1}{\mathrm{n}}+\frac{2 \mathrm{i}}{\mathrm{n}^{2}}+\frac{\mathrm{i}^{2}}{\mathrm{n}^{3}}=\lim _{\mathrm{n} \rightarrow \infty} 1+\frac{1}{\mathrm{n}}+1+\frac{1}{3}+\frac{1}{2 \mathrm{n}}+\frac{1}{6 \mathrm{n}^{3}}=1+0+1+\frac{1}{3}+0+0=2 \frac{1}{3}$.
The area under the curve $y=x^{2}$ from $x=1$ to $x=2$ is $2 \frac{1}{3}$. Like for most Riemann sum problems, finding the answer was long and tedious. In the next section we'll see how much easier it is to solve this same problem by using integrals.

## Integrals

The Fundamental Theorem of Calculus, Part I:

The function $f(x)$ is the derivative of the area underneath it: $\frac{d A}{d x}=f(x)$

To find the area A, we use the antiderivative of $\mathrm{f}(\mathrm{x})$.

The Fundamental Theorem of Calculus, Part II:
$\int_{a}^{b} f(x) d x=F(b)-F(a) \quad$ where $F(x)$ represents the antiderivative of $f(x)$.

Note that $\int_{a}^{b} f(x) d x=-\int_{b}^{a} f(x) d x$.

For many years I thought that poor Mr. Riemann was going to all that trouble to calculate Riemann Sums because he didn't know about integrals. However, Bernhard Riemann was born in 1826, which is just about an entire century after the death of Isaac Newton! What Riemann Sums do is help us understand integrals and see them as a limit (as $n$ goes to infinity) of the sum of the areas of $n$ rectangles. At this point you might also suspect that Professor Riemann had an ulterior motive, since a hefty dose of Riemann Sums is sure to make any student appreciate integrals.

So, how were integrals discovered? The story of integrals actually began with infinitely small differences.

We know we want the area under a curve. When we say "under" we mean the area between the curve and the $x$-axis. If the function is positive that area will be positive. When the function value is negative the area we are interested in is below the x -axis, and it is considered to be a negative area.


Now that we are clear on that, let's actually find the area under a curve. Using the principles of calculus, we consider how the area grows. That is, we want to look at dA, an infinitely small increase in the area under the curve. How is dA related to an infinitely small increase in $x$, the distance along the $x$ axis? If you have worked with Riemann Sums for a bit, you should be able to see that as $x$ increases by an infinitely tiny amount $d x$, the area under the curve increases by an infinitely tiny strip with an area equal to $f(x)$ times $d x$ :
$d A=f(x) d x$
In the picture below, dA is represented by a thin blue line:


The next image shows the detail of $d A$, the infinitely thin strip that represents the increase in the area. The width of the strip is $d x$. The length on the left side is $y$, which is $f(x)$, and on the right it is $\mathrm{y}+\mathrm{dy}$. Back in the section "The Idea of Calculus", we said that because dy is infinitely small, $y+d y=y$, so both lengths are the same. As a result, we can calculate that the area of each strip is $y d x$, or $f(x) d x$.


We also said that if you cut something up into infinitely many infinitely tiny pieces you can put it back together by taking the sum of those pieces. Therefore $A$, the area we want, is equal to $\int \mathrm{dA}$. Recall that the fancy symbol $\int$ is just the letter $S$ and it stands for Sum, in this case the sum of all the infinitely tiny dA's. This sum is called an integral. All we are doing here is dividing the area under the curve up into infinitely tiny strips and then adding the areas of all of the strips together. Because $d A=f(x) d x, \int d A$ should be equal to $\int f(x) d x$. So now we have:
$d A=f(x) d x$
$\int d A=\int f(x) d x$
$A=\int f(x) d x$
Well that's nice, but $f(x)$ is probably different at every spot along the curve. This looks like the worst Riemann sum of all time, since there are infinitely many intervals $d x$ and the area of the rectangle for each one would have to be calculated. That seems impossible, but the good part is that the final result would also be the most accurate Riemann sum ever. In fact, it would be perfectly accurate.

Actually, it is possible to calculate this sum, and it isn't even hard to do. The key is that $d A=f(x) d x$. That means that $\frac{d A}{d x}=f(x)$. It may not jump out at you when it is written in symbols like this, but this actually says that $f(x)$ is the derivative of the area underneath itself!


That must have been very exciting for the people who first discovered it, because it provides an amazingly easy way to find the area underneath a curve. If we know that $f(x)$ is the derivative of the area (the rate of change of the area), then all we have to do to get the actual area is to find an antiderivative of $f(x)$. That is, we find the function that has $f(x)$ as its derivative.
$A=\int f(x) d x=$ the antiderivative of $f(x)$
Let's see how this works. In particular, we might want to find the area under the parabola $y=x^{2}$ which was a major headache for mathematicians to obtain before the discovery of calculus. In this case $f(x)=x^{2}$, so the derivative of the area under this curve is $x^{2}$. How can we make a function that has the derivative $x^{2}$ ? A little trial and error will tell you that $F(x)=\frac{1}{3} x^{3}$ would do the job. We say that $\frac{1}{3} x^{3}$ is the antiderivative of $x^{2}$. The antiderivative is often indicated by using a capital $F$ as opposed to the lowercase $f$ we usually use for functions. So there you are; the area under the curve can be calculated by using this function:
$A=\frac{1}{3} x^{3}$.
$f(x)$ is the derivative of the area underneath its graph, and therefore the area under the graph is the antiderivative of $f(x)$. This discovery is so important that it has been designated as the Fundamental Theorem of Calculus, part I. Sadly, most calculus courses fail to convey how exciting this is (or even how anyone might find calculus exciting in any way), but it is really
amazing that we can find the exact areas underneath curves, and it has many useful practical applications.

Since $A=\frac{1}{3} x^{3}$, when $x=0$ the area is 0 , and when $x=1$ the area is $\frac{1}{3}$. So, the area between 0 and 1 is $\frac{1}{3}$. If I wanted the area between 0 and 2 , I would plug 2 into the formula and get $A=\frac{8}{3}$. This simple new method turned out to be useful to find many quantities that can be represented by the area under a curve, and it resulted in many important advances in science and technology.

The Fundamental Theorem of Calculus makes sense when you think about it in terms of position and velocity. For example, consider this graph:


If you walked for 4 hours, at a leisurely speed of 2.5 miles per hour, you would cover a distance of 10 miles. The area under the curve represents the distance covered (the change in your position). As we just saw, we can find this area by using an antiderivative. By looking at the graph you can see that the function pictured is $f(t)=2.5$, where $t$ is the time and $f(t)$ represents the speed. It is easy to find something that has 2.5 as its derivative: 2.5 t would do nicely. So the area (as a function of time) can be represented by the formula $A=2.5 t$. When $t=4$ the area is 10 , as expected.

An average speed of 2.5 miles per hour would allow you to cover the same distance:


Looking at this graph, I would say that the slope of the line is $5 / 4 . f(t)=\frac{5}{4} t$ would be the function that represents the speed in this case. Simple geometry tells you that the area under the graph is $1 / 2$ the base times the height of the triangle, which is 10 (miles). To get the same result using calculus, you should find an antiderivative for $\frac{5}{4}$ t. $\frac{5}{4} \mathrm{t}^{2}$ doesn't quite fit because its derivative is too big. $\operatorname{Try} \frac{5}{8} \mathrm{t}^{2}$ and you should find that it is just right. When $\mathrm{A}=\frac{5}{8} \mathrm{t}^{2}$ the area will be 10 when $t=4$.

Regardless of the shape of a velocity curve, the area underneath it represents the net change in position. However, because an area under the $x$-axis is considered negative, the change in position may not represent the total distance traveled.

Just as the velocity is the derivative of the position curve, the position function is the antiderivative of the velocity curve.

Now, how would you find the area between $x=1$ and $x=2$ underneath the curve $y=x^{2}$ ? Stop and think about it for a bit before you continue reading.

Using the formula $A=\frac{1}{3} x^{3}$ you would probably calculate it like this:
$\frac{8}{3}-\frac{1}{3}=\frac{7}{3}$. In fact, if you wanted to know the area between any two values of $x$, say $x=a$ and $x=b$, you would calculate the area at $b$ and subtract the area at $a$. The area between $a$ and $b$ would be
$\frac{1}{3} b^{3}-\frac{1}{3} a^{3}$.
This idea, which you could easily come up with yourself, is the Fundamental Theorem of Calculus, part II. $F(x)$ is the antiderivative of $f(x)$, and $F(x)$ is the function that represents the area underneath $f(x)$. If we want the area between $a$ and $b$, we find it as follows: $F(b)-F(a)$.

Since we will often want the area between two specific points we add them to our integral notation, so we write the area from point $a$ on the $x$-axis to point $b$ on the $x$ - $a x i s$ as $\int_{a}^{b} f(x) d x$. $\int_{a}^{b} f(x) d x=F(b)-F(a) \quad$ where $F(x)$ represents the antiderivative of $f(x)$, and $f(x)$ is the derivative of $F(x)$.

If you look closely at the narrative above, you may see that I omitted something. I presented $\frac{1}{3} x^{3}$ as the antiderivative of $x^{2}$. Actually, there is another antiderivative that would do the same job. Consider $F(x)=\frac{1}{3} x^{3}+4$. The derivative of this function is also $x^{2}$. Can we describe the area under the curve using this function instead? Well, let's try it out.
$A=\frac{1}{3} x^{3}+4$
When $x=0, A=4$. Hey, wait a minute, how can that be 4 when it looks like it should be 0 from the graph? You can think of this as an "initial condition". Suppose some object has already traveled a distance of 4 units, and from here on it will travel a distance given by the area underneath $\mathrm{y}=\mathrm{x}^{2}$. So, when $\mathrm{x}=1, \mathrm{~A}=4+\frac{1}{3}$. Between $\mathrm{x}=0$ and $\mathrm{x}=1$ the area would be $4 \frac{1}{3}-4$, which is still $\frac{1}{3}$. In fact, any function $A=\frac{1}{3} x^{3}+C$, where $C$ is a constant, would allow us to calculate the area below the curve. $\frac{1}{3} x^{3}+C$ is the general antiderivative of $x^{2}$. To make sure they include all of the possibilities, mathematicians write $\int f(x) d x=F(x)+C$ for a general integral with no specific endpoints.

For integrals between two specific points, like $\int_{a}^{b} f(x) d x$, the constant $C$ cancels out: $F(b)+C-$ $(F(a)+C)=F(b)-F(a)$. After having to remember to write C's every time you solve an integral, you'll be happy to get rid of C . For future reference though, you need to understand that C does not actually disappear. The initial condition represented by C in a real-life situation is always there, but it doesn't matter when you look at the cumulative result of changes that happen between two points. Well, maybe "doesn't matter" isn't exactly right. If you have already walked 10 miles, and then I use an integral to calculate that you walked an additional mile between an arbitrarily set time $t=0$ and $t=1,10$ miles cancels out but you'll still feel like you walked 11 miles.

Try all this out with a simple problem like "find the area underneath $f(x)=5$, from $x=3$ to $x=7$. This problem doesn't require calculus at all, so do it first without, and then with calculus. Use simple geometry to get the area from 0 to 7 , and then the area from 0 to 3 . Subtract the two values to get the area between 3 and 7 , which should be 20.

In calculus notation it works like this:
The antiderivative of 5 is $5 x$ (or $5 x+C$ if you are just considering that in a general way). To find the area between 3 and 7 , we take the area at $x=7$ and subtract the area at $x=3$ :
$\int_{3}^{7} 5 \mathrm{dx}=5 \mathrm{x}$, evaluated at 7 , minus 5 x evaluated at 3 . That works out to $5 \cdot 7-5 \cdot 3=20$. The shorthand notation for this is $5 x\left[\begin{array}{l}7 \\ 3\end{array}=5 \cdot 7-5 \cdot 3=20\right.$.

If you understand how this works, you can also see that $\int_{7}^{3} 5 \mathrm{dx}=-20$. The area isn't negative in a real sense, but we are looking in the opposite direction (from a larger $x$ to a smaller $x$ ) so we get a negative value. In general, $\int_{a}^{b} f(x) d x=-\int_{b}^{a} f(x) d x$.

Another way to get a negative value for an integral occurs when the area under the function graph is below the $x$-axis. Let's find the area under $f(x)=-5$ from $x=3$ to $x=7$ :
$\int_{3}^{7}-5 d x=-5 x$, evaluated at 7 , minus $-5 x$ evaluated at 3 . That works out to $-5 \cdot 7--5 \cdot 3=$ -20. Again, the shorthand notation for this is $-5 x\left[\begin{array}{l}7 \\ 3\end{array}=-5 \cdot 7--5 \cdot 3=-20\right.$. As expected, the area turns out to be negative.

## Example

Find the area under the line of the function $f(x)=2 x$, from $x=0$ to $x=5$, using both calculus and Riemann sums. Check your work using geometry.

First, let's use calculus because it is the easiest here. The area is the integral, from 0 to 5 , of the function $2 x$. Each infinitely small piece of the area is $d x$ wide, and $f(x)$ tall: $\int_{0}^{5} f(x) d x$. For this particular function the integral is $\int_{0}^{5} 2 x d x$. The antiderivative of $2 x$ is $x^{2}$, and we have to evaluate that at 0 and $5: \int_{a}^{b} f(x) d x=F(b)-F(a)$ so $\int_{0}^{5} 2 x d x=5^{2}-0^{2}=25$.

Next, sigh, Riemann sums. You may think that you don't need them anymore, but they are still on the AP exam so you can't forget them. The width of the interval is 5 , and we are dividing that up into n intervals. Each interval is $\frac{5}{\mathrm{n}}$ wide, and when n is infinitely large that will correspond to dx . We have to find the height of each rectangle at the end of each interval, so we have to plug the $x$ value at that point into the function $f(x)=2 x$. The end of the first interval is at $x=\frac{5}{n^{\prime}}$ and the end of the $i^{\text {th }}$ interval is at $\frac{5 i}{n}$. At that point the function value is $2\left(\frac{5 i}{n}\right)$. Now add up all of the intervals, and take the limit as $n$ goes to infinity:
$\lim _{n \rightarrow \infty} \sum_{i=1}^{n} 2\left(\frac{5 i}{n}\right) \cdot \frac{5}{n}$
Notice that it is this expression that corresponds to $\int_{0}^{5} 2 \mathrm{xdx}$.
$\lim _{\mathrm{n} \rightarrow \infty} \sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{10 \mathrm{i}}{\mathrm{n}} \cdot \frac{5}{\mathrm{n}}=\lim _{\mathrm{n} \rightarrow \infty} \sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{50 \mathrm{i}}{\mathrm{n}^{2}}=\lim _{\mathrm{n} \rightarrow \infty} 50 \sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{\mathrm{i}}{\mathrm{n}^{2}}$
Recall that $\sum_{i=1}^{n} i$ means the sum of the first $n$ numbers, which is always $\frac{n}{2}(1+n) \cdot \sum_{i=1}^{n} \frac{i}{n^{2}}$ is
$\frac{\frac{n}{2}(1+n)}{n^{2}}$. Simplify like this: $\frac{\frac{n+n^{2}}{2}}{n^{2}}=\frac{n+n^{2}}{2 n^{2}}=\frac{1}{2 n}+\frac{1}{2}$.
$\lim _{\mathrm{n} \rightarrow \infty} 50 \sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{\mathrm{i}}{\mathrm{n}^{2}}=\lim _{\mathrm{n} \rightarrow \infty} 50\left(\frac{1}{2 \mathrm{n}}+\frac{1}{2}\right)=25$.
Geometry tells us that our answers are correct, because the area under $f(x)=2 x$ from $x=0$ to $x$ $=5$ is a triangle with a base of 5 and a height of $f(5)$ which is 10 . Area $=\frac{1}{2} \cdot 5 \cdot 10=25$.

## Integral Basics

$\int_{a}^{b} f(x) d x$ is a definite integral that will have a numeric value.
$\int f(x) d x$ is an indefinite integral. Add a constant $C$ when you integrate.

$$
\int c f(x) d x=c \int f(x) d x
$$

The integral of a sum is the sum of the separate integrals: $\int\left(x^{2}+x\right) d x=\int x^{2} d x+\int x d x$.

$$
\int_{a}^{b} f(x) d x=\int_{a}^{c} f(x) d x+\int_{c}^{b} f(x) d x \text {, even if } c \text { is not in the interval }[a, b]
$$

Now that you know what integrals are, your textbook will have you jump right in and start solving them. Before you do that, let's look at a few basic rules.

At the beginning of this e-book, we saw that $\int \mathrm{dx}=\mathrm{x}$. You can think of that as dividing some length $x$ up into infinitely many, infinitely tiny pieces of length $d x$. The length $x$ is a changing quantity, and it grows by adding infinitely tiny amounts dx . When you add up all of the pieces of size $d x$, you get $x$ back. A different way to think of that is to represent $x$ as the area under a function graph.

Consider the constant function $\mathrm{y}=1$. The area under the graph from $\mathrm{x}=0$ to $x=7$ is $7 \cdot 1=7$. That is written as $\int_{0}^{7} 1 d x$, or $\int_{0}^{7} d x$. In this case you have already decided from where to where you want the area so you can specify the lower and upper limits of integration, 0 and 7 , and you get a definite integral that will have numerical value. In general, we can find the area by using the integral $\int 1 \mathrm{dx}$, which is called an indefinite integral. If you are solving an indefinite integral, don't forget to add in a constant, because the derivative of something like $2 x$ +5 is the same as the derivative of $2 x . \int 2 d x=2 x+C$.

The method of calculus divides an area up into infinitely many infinitely thin strips. For the area under $f(x)=1$, each strip will have a height of 1 and a width of $d x$. The sum of all of these strips will be $\int 1 d x$, which is equal to $x(+C)$. Notice that the derivative of the area function is the function $f(x)=1$. The area function is the antiderivative of $f(x)=1$.

Now, suppose that we wanted to find the area under $f(x)=3$. We can again divide that up into strips with a width of $d x$, but now each strip will be 3 times as tall. The sum is now $\int 3 d x$, which is equal to $3 x(+C)$. Notice that this integral is 3 times as large as $\int 1 \mathrm{dx}$ :
$\int 3 d x=3 \cdot \int 1 d x$
In general, if you divide the area under any function graph up into strips of height y and width $d x$, you would expect that if you make all those strips three times as tall you would get three
times the area. This means that you can take a constant that is inside the integral, and put it outside:
$\int c f(x) d x=c \int f(x) d x$.
If you stop and think about that, you can see that it is really using factoring:
$3 y_{1} d x+3 y_{2} d x+3 y_{3} d x+3 y_{4} d x+\ldots=3\left(y_{1} d x+y_{2} d x+y_{3} d x+y_{4} d x+\ldots\right)$

It is also possible to find an integral of a function that contains a sum. For example, $y=x^{2}+x$ can be seen as a sum of two separate functions, $f(x)=x^{2}$ and $g(x)=x$. Just as you can take the derivative of these separate parts and add them up to get the derivative of the whole, you can get the integral of the sum by adding up the separate integrals:
$\int\left(x^{2}+x\right) d x=\int x^{2} d x+\int x d x$
That makes sense, because every $y$ value of $y=x^{2}+x$ is really composed of the two $y$-values of each function stacked on top of each other. $\int\left(x^{2}+x\right) d x=\frac{1}{3} x^{3}+\frac{1}{2} x^{2}+C$.
$\int(f(x)+g(x)) d x=\int f(x) d x+\int g(x) d x$.
Also, $\int(f(x)-g(x)) d x=\int f(x) d x-\int g(x) d x$.
It would be very nice if $\int(f(x) \cdot g(x)) d x$ would be equal to $\int f(x) d x \cdot \int g(x) d x$, but unfortunately that is not the case. Just like when we tried to find $\mathrm{d}(\mathrm{uv})$ at the beginning of this book, a little math is required to come up with the right answer. We'll worry about that later, once you are more used to finding integrals.

## Using a Graphing Calculator to Find an Integral

To find a definite integral on the TI-84 calculator, go to the MATH menu and look for fnInt(.

For the TI-83 and TI-84 calculator, you can go to the MATH menu and look for fnInt(. On the
newer models you can just fill in the blanks on the integral that appears. On the older models, enter the function, followed by a comma, followed by $x$, another comma, and the upper and lower limits of integration, separated by commas. Then close the function parentheses and press ENTER.

Online integral calculators are readily available. Don't hesitate to use them to check your work!

## The Area of a Circle

The area of a circle is $\int 2 \pi r d r=\pi r^{2}$.

The formula for the circumference of a circle was discovered in ancient times when people wondered how many times the diameter of a circle would fit around the circumference. They named this number $\pi$, and said that the circumference of a circle is $\pi D$, where $D$ is the diameter. This is where we get our formula $C=2 \pi r$.

If we did not know how to find the area of a circle we could use calculus to discover the formula. If the radius of a circle increases by an infinitely tiny amount dr, the infinitely tiny increase in the area of the circle, dA , would be the area of an infinitely tiny ring around the circle with a thickness of dr. Normally you can't quite change a ring into a rectangular strip, because the inner edge of the ring is smaller than the outer edge. However, when you have an infinitely thin ring things are different. The circumference of the ring at the inner edge is $2 \pi r$, and at the outer edge it is $2 \pi(r+d r)$. Since $r+d r=r$, both the inner and the outer edge have the same length. We can turn the ring into a strip with width $d r$ and length $2 \pi r$. That strip, which represents the infinitely small change in the area, $d A$, has an area of $2 \pi r \cdot d r$. $d A=2 \pi r d r$

Now we can take the sum of all the tiny dA 's to get the whole area of the circle:
$A=\int d A=\int 2 \pi r d r$.
$\int 2 \pi r d r$ is the antiderivative of $2 \pi r$, for which the simplest solution is $\pi r^{2}$. When we look at integrals as functions we will see that this is the only solution.

## Splitting an Integral into two Parts

$$
\int_{a}^{b} f(x) d x=\int_{a}^{c} f(x) d x+\int_{c}^{b} f(x) d x \quad c \text { does not need to be between } a \text { and } b
$$

Areas can easily be split up into two parts, and you would not be surprised to find that the area represented by $\int_{3}^{7} 5 \mathrm{dx}$ could be written as $\int_{3}^{4} 5 \mathrm{dx}+\int_{4}^{7} 5 \mathrm{dx}$. What may be a little harder to imagine is that we can split the same integral up like this: $\int_{3}^{7} 5 \mathrm{dx}=\int_{3}^{20} 5 \mathrm{dx}+\int_{20}^{7} 5 \mathrm{dx}$. That seems really strange because 20 is not even in the interval [3,7]. However, when you calculate it out you will find that the total area is $85+(-65)$ which is still 20.

## Absolute Value Functions

$\int_{a}^{b}|f(x)| d x$ may differ in value from $\int_{a}^{b} f(x) d x$.

Use a graphing calculator, or manually account for where $f(x)$ changes signs.

Absolute value functions take the negative part of the corresponding regular function, and make it positive. You cannot expect an integral to automatically account for this, although a graphing calculator will. If you are permitted to use a calculator to find your integrals, you can actually use the absolute value function of your calculator inside the integral function and have it find the correct answer for you.

There is no clever way to just find the area underneath say $y=|2 x-3|$ from $x=0$ to $x=5$ without a graphing calculator. You have to figure out where the absolute value sign is changing the function from a negative value to a positive value, and then calculate the two areas separately. The illustration below shows the area you want to calculate colored in green. The red area is what will be included in your integral instead if you are not careful to account for the absolute value.

$|2 x-3|$ is 0 when $2 x=3$, so the function value is zero when $x$ is 1.5 . When $x$ is less than 1.5 the function would be negative, but the absolute value sign makes it positive. If you calculate the integral from 0 to 1.5 of $2 x-3$, you get $\int_{0}^{1.5}(2 x-3) d x=\left(x^{2}-3 x\right)\left[\begin{array}{c}1.5 \\ 0\end{array}=2.25-4.5-0=-2.25\right.$.

That is a negative value, because the area is below the $x$-axis. Because you know you are dealing with an absolute value function and the area will be above the $x$-axis instead, you should take the negative integral:
$-\int_{0}^{1.5}(2 x-3) \mathrm{dx}=2.25$. Notice that the area marked in red in the figure actually is $2.25(1 / 2$ of 3 times 1.5). The larger area marked in green is a triangle with base 3.5 and height 7 , so it should have an area of $\frac{1}{2} \cdot 3.5 \cdot 7=12.25$. $\int_{1.5}^{5}(2 x-3) \mathrm{dx}$ is 12.25 .
$\int_{0}^{5}|2 x-3| d x=2.25+12.25=14.5$
Absolute value integrals are particularly important for some applications. For example, when you are looking for the distance traveled by some object, you may just think: well, distance = velocity • time. If the velocity is not constant, l'll just take the velocity during one small time interval dt , and then create an integral. The distance traveled is the area under the velocity curve. Unfortunately, velocity can be negative. In this case the distance traveled is given by the absolute value of the area under the velocity curve. When you are looking for distance traveled it is easiest to always just consider the absolute value of the velocity. Make a habit of using an absolute value sign in your integral when working with velocity.

## Piecewise Defined Functions

You may be asked to find the integral of a piecewise defined function. Even if such a function has a jump discontinuity you can find the areas under each part separately and add them up. Due to the recent insertion of this topic into the AB curriculum that may not make much sense to you as one of the integrals doesn't have a defined limit on one side. That little problem is normally explained away in the BC course.

## Even and Odd Functions

For even functions, it is often convenient to find half the integral and then double it.

Odd functions, where $f(-x)=-f(x)$, will have a zero value integral from -a to a.

Now that you know how to find the area underneath $y=x^{2}$, you can calculate the area under this curve from -1 to 1 . That is equivalent to finding the value of the integral $\int_{-1}^{1} x^{2} d x$. You should find that this area is $\frac{2}{3}$, or exactly double the area from 0 to 1 that we calculated already. When you look at the graph you can see why: $y=x^{2}$ is symmetric about the $y$-axis. Many functions are symmetrical like this so it is often faster to calculate half the area (starting at 0 ) and then double it. You can recognize even functions, because they all have the property that $f(x)=f(-x)$. Just put $-x$ into the function in place of $x$ and see if you get the same result.

Next, calculate the value of $\int_{-2}^{2} x^{3} d x$. You should end up with a value of 0 . If you look at the graph of $y=x^{3}$ you can see the reason for that: $f(x)=x^{3}$ is an odd function. The area below the $x$-axis is exactly equal to the area above the $x$-axis so the two values cancel each other out. If you rotate the graph around the origin 180 degrees, it looks exactly the same. Odd functions are also easy to spot because $f(-x)=-f(x)$. For an odd function, the area under the curve from -a to a is always 0 . Remember that because it can save you a lot of work.

## VIII. The Integral as a Function

The "Area so Far" function is $A(x)=\int_{a}^{x} f(t) d x$.
We have to use the letter $t$ because we can't use $x$ twice to do two different jobs.
$t$ is a dummy variable that doesn't appear in the final result: $\int_{a}^{x} f(t) d x=F(x)-F(a)$.
$\int_{0}^{1} 2 x d x \quad \int_{0}^{2} 2 x d x \quad \int_{0}^{3} 2 x d x \quad \int_{0}^{4} 2 x d x \quad \int_{0}^{5} 2 x d x \quad \int_{0}^{6} 2 x d x$
These integrals form a steady progression. Each integral represents a larger area than the last, and that value increases as we increase the number that makes up the upper bound of the integral. Hmm, I could turn that into a function if I used a variable for the upper bound. That way I could have a function that relates the area A to the upper bound. Now, what variable should I pick for the upper bound? Well, it is increasing along the $x$-axis, so I would probably pick x :
$A(x)=\int_{0}^{x} 2 x d x$
After all, that says what I want, the integral from 0 to $x$ of $2 x$. Or does it??
Unfortunately there is a bit of a problem with the variable $x$ here. It is quite fine to stick $x$ in an equation or expression more than once, but once you pick a value for $x$ that value should be the same everywhere. For example, in the equation $x^{2}-5 x+6=0$ you can select $x=3$ or $x=2$, but you can't pick 3 for the first $x$ and 2 for the second one. You also can't pick 3 for the first $x$ and just leave the second $x$ there. So, if I am calculating the area under $2 x$, and $I$ select $x=7$ for the upper bound, $I$ should get $A(7)$, the area from 0 to 7 under $2 x$. But oops, now that $x=7,2 x$ doesn't just stay there. It turns into 2(7) which is 14:
$A(7)=\int_{0}^{7} 2(7) d x=\int_{0}^{7} 14 d x$

If you compare the value of this integral with the intended one, $\int_{0}^{7} 2 \mathrm{xdx}$, you will see that it is not the same at all.

Let's try again. This time I will pick $t$ for the variable of the upper bound, so this will be a function of $t$ :
$A(t)=\int_{0}^{t} 2 x d x$
That should work, since I can now set $t$ equal to any number without affecting 2 x :
$A(t)=\int_{0}^{t} 2 x d x=x^{2}\left[\begin{array}{l}t \\ 0\end{array}=t^{2}-0=t^{2}\right.$
Since this does work, I can start turning it into a more general function, using $f(x)$ instead of the particular function 2 x for the part inside the integral:
$A(t)=\int_{0}^{t} f(x) d x$
Although this is workable and mathematically correct, there is a minor problem with the aesthetics of it. $A(t)$ is a function of $t$, and all of our other functions are functions of $x$. Physicists like to see things as a function of time, but mathematicians have a strong preference for functions of $x$, possibly because that just fits better into an $x-y$ coordinate system. Well, not to worry; a rose by any other name would smell as sweet. We can just turn the variables around without changing the mathematical facts:
$A(x)=\int_{0}^{x} f(t) d x$
$A(x)$ is often called "the area so far" function.
This is the official formula that goes into the textbooks, while everything else was the rough draft that ended up in the trash. Your textbook may present this final version rather casually without much of an explanation as to where $t$ came from. Don't be deceived, because the difference between $t$ and $x$ suddenly matters a lot when test time comes around.

When we solve $A(x)=\int_{0}^{x} f(t) d t$, we look for the antiderivative of $f(t)$, which will be $F(t)$, and evaluate that at $t=x$ and $t=0$ to get $A(x)=F(x)-F(0)$. Notice that having an $x$ in the upper bound doesn't affect how the integral is evaluated. It still works just like $\int_{a}^{b} f(x) d x=F(b)-F(a)$, according to the Fundamental Theorem of Calculus.

Because $t$ does not appear in the final result, it is a considered a placeholder variable, sometimes called a "dummy variable".
$A(x)$, the "area so far" function, can also extend to negative values of $x$. For $A(x)=\int_{0}^{x} f(t) d t$, if $x$ is smaller than 0 we are looking in the opposite direction, so if the area under the curve is positive over this interval, $A(x)$ will actually have a negative value.

It also does not matter if we actually start at 0, or at some other number:
$\int_{-1}^{1} 2 x d x \quad \int_{-1}^{2} 2 x d x \quad \int_{-1}^{3} 2 x d x \quad \int_{-1}^{4} 2 x d x \quad \int_{-1}^{5} 2 x d x \quad \int_{-1}^{6} 2 x d x$
This sequence of integrals also represents an orderly change in the area, just like when we started at 0 . We can represent the "area so far" function as $A(x)=\int_{a}^{x} f(t) d t$, where $a$ is any constant.

## Example

Let's take a look at a dummy variable in action. We will find the proper function for the area of a circle, which should look like $A(r)=\pi r^{2}$. Here you should note that we want a general function, not the area of a specific circle with a pre-determined radius. First, we'll create a circle with a radius $r$, which is 6 in the illustration below. The variable $r$ will represent the upper bound of the integral, which is the radius of the circle. The function $A(r)$ will return the area for any value of $r$. However, during the actual process of integration $r$ is not really changing.


To find the area, we divide the circle up into infinitely thin rings, one of which is shown in light blue in the picture above. The circumference of this ring is $2 \pi \ldots$... oh, we can't put $r$ here because that is the radius of the whole circle, which is 6 right now. We also can't just say that the radius of this infinitely thin ring is 3 , because we will need to create infinitely many of these rings, each with a different radius. I guess we are going to need a different variable, which can't be called r. Let's call it q. Now each infinitely thin ring will have an area of $2 \pi q$ times d.... Again we have to stop and think carefully what to put here. The radius of the light blue ring is $q$, so if that increases by an infinitely tiny amount we would call that amount dq. $d A$, the infinitely small area of this ring is $2 \pi q$ dq. Once we have that we can put all our infinitely tiny rings together to get the area of the whole circle. The integral starts at 0 , the center of the circle, and goes all the way to $r$, which is 6 in this particular case but could be any positive value. $q$ will be the variable of integration:
$A(r)=\int_{0}^{r} 2 \pi q d q$
That would be $\pi q^{2}$, evaluated at $r$ and $0: \pi r^{2}-0=\pi r^{2}$
$A(r)=\int_{0}^{r} 2 \pi q d q=\pi r^{2}$
You can plug any value for $r$ into this function, which means that $r$ is a real variable. As $r$ increases, so does the value of the integral, because its upper bound increases. $q$ is the dummy
variable that does all the hard work inside the integral but gets none of the credit.

## The Derivative of the Integral Function

If $F(x)=\int_{a}^{x} f(t) d x$, then $F^{\prime}(x)=f(x)$.
$F^{\prime}(x)=\lim _{h \rightarrow 0} \frac{F(x+h)-F(x)}{h}$. You could write that as $\lim _{h \rightarrow 0} \frac{\int_{x}^{x+h} f(t) d t}{h}$.

A favorite test question is:
$A(x)=\int_{0}^{x} f(t) d t$, what is the derivative of $A(x)$ ?
You might think that you'll be allowed to be just as casual about these variables as your textbook and answer $\mathrm{f}(\mathrm{t})$, but you'd be wrong. If you care about your grade or AP score, answer $f(x)!A(x)$ is a function of $x$, and its derivative will also be a function of $x$. There is no reason to put the dummy variable into the derivative, since it is only there temporarily to help us find the actual integral.

We just saw that $A(r)=\int_{0}^{r} 2 \pi q d q=\pi r^{2}$, so what is $A^{\prime}(r)$ ? Well, that is just $2 \pi r$, not $2 \pi q$.

The derivative of an integral function is the same for any fixed lower bound of the integral, not just for $x=0$. Take for example, $A(x)=\int_{0}^{x} 2 t d t . A(x)$ is the antiderivative of $2 t$, which is $t^{2}$, which needs to be evaluated at $x$ and 0 . That gives $A(x)=x^{2}-0=x^{2}$. The derivative of that is $2 x$. Next, consider $A(x)=\int_{5}^{x} 2 t d t$, which is $t^{2}\left[\begin{array}{l}x \\ 5\end{array}=x^{2}-25\right.$. Take the derivative of that, and it is
still $2 x$. If you look at the pictures below, you can see that it doesn't make a difference where the lower bound is. The increase in area is still the infinitely thin light blue line, and the area of that is $d A=f(x) d x$. Divide both sides by $d x$ to get the derivative: $\frac{d A}{d x}=f(x)$.



Now we can see that for any constant $a, A(x)=\int_{a}^{x} f(t) d t$ has the derivative $f(x)$.

## U-Substitution for the Integral Function (Have integral, would like to trade for derivative.)

To find the derivative of $A(x)=\int_{a}^{x^{2}} f(t) d t$, let $x^{2}=u: \quad A(x)=\int_{a}^{u} f(t) d t$ $\frac{d A}{d u}=f(u)$. Use the chain rule: $\frac{d A}{d x}=\frac{d A}{d u} \cdot \frac{d u}{d x}=f(u) \cdot 2 x=f\left(x^{2}\right) \cdot 2 x$.

Once someone wrote the equation $A(x)=\int_{a}^{x} f(t) d t$, there was really nothing to stop someone else, or perhaps the same person, from writing the equation $A(x)=\int_{a}^{x^{2}} f(t) d t$. That's nice, but it may be a bit hard to visualize what this means.

To get a better idea of what is going on we can look at a simple example: $A(x)=\int_{0}^{x^{2}} 5 d t$.
If $x=3$ here, and $f(t)$ is 5 , then $A(x)$ represents the area under $f(t)$ from 0 to 9 , which is 45 . When $x$ is $4, A(x)=16 \cdot 5=80$. In general, $A(x)=F\left(x^{2}\right)-F(0)$. Since $F$ here is $5 t$, this "area so far" function is given by $A(x)=5 x^{2}-0$. While $f$ is a linear function, the Area function is quadratic.

If you were to ask, "How fast is $\mathrm{A}(\mathrm{x})$ increasing as $\mathrm{x}^{2}$ increases?" the answer would still be 5 . That is, the derivative of $A(x)$ with respect to $x^{2}$ is 5 . In this simple example the rate of change of the area is constant at every point $x^{2}$, which allows us to see exactly what is happening. If $x^{2}$ is 9 the area is 45 , and if $x^{2}$ is 10 the area is 50 . The area is increasing by 5 units for every unit increase in $x^{2}$.

But, as you might guess by now, mathematicians will ask for the derivative of $A$ with respect to $x$, which is asking: how fast is $A(x)$ increasing per unit increase in $x$ ? As you can see, when $x$ increases by one unit from 3 to $4, A$ increases by 35 . When $x$ increases from 4 to 5 , we get $A(4)=80$ and $A(5)=125$, which is an increase of 45 . This tells us that the rate of change with
respect to $x$ is not constant; in fact it is increasing. That is not surprising as $A(x)$ is a quadratic function. Because we have chosen a simple function for $f(x)$, we can easily find $\int_{0}^{x^{2}} f(t) d t$ and then take the derivative:
$\int_{0}^{x^{2}} 5 d t=5 t\left[\begin{array}{c}x^{2} \\ 0\end{array}=5 x^{2}-0=5 x^{2}\right.$. The derivative of $5 x^{2}$ is just $10 x$, so we're done. Unfortunately, there are many integrals that are not so easy to evaluate. For example, $\int_{1}^{x^{2}} \frac{1}{4-t^{2}} \mathrm{dt}$ is a bit hard to evaluate because it is not immediately obvious what the antiderivative is. In a case like this, if you only need the derivative why bother finding the antiderivative at all?

To illustrate how to get the derivative of the area function without finding the antiderivative, we will use the function $A(x)=\int_{0}^{x^{2}} \sqrt{t} d t$. We know how to get the derivative of $A(x)=\int_{0}^{x} f(t) d t$, which is $f(x)$. To make our function look like that, we substitute $u$ for $x^{2}$, to get $A(u)=\int_{0}^{u} \sqrt{t} d t$. . Notice how the area function is now called $A(u)$, but the picture is still the same:


By looking at the picture, we can see that a very small change in the area, $d A$, is now $f(u) d u$.
We already know that $u=x^{2}$, and $d u=2 x d x$, so the answer is actually visible in the picture. $f(u)$ will be $\sqrt{\mathrm{x}^{2}}$, which is $|\mathrm{x}|$, so the area of the blue strip is $|\mathrm{x}| \cdot 2 \mathrm{xdx}$. That makes the derivative equal to $|x| \cdot 2 x$.
$d A=f(u) d u$
$d A=f\left(x^{2}\right) \cdot 2 x d x$
$d A=\sqrt{x^{2}} \cdot 2 x d x$
$\frac{\mathrm{dA}}{\mathrm{dx}}=|\mathrm{x}| \cdot 2 \mathrm{x}$
It's a bit of a pain to draw a picture each time, so you may prefer to use the chain rule:
$A(u)=\int_{0}^{u} \sqrt{t} d t$
$\frac{d A}{d x}=\frac{d A}{d u} \cdot \frac{d u}{d x}$
$\frac{\mathrm{dA}}{\mathrm{dx}}=\sqrt{\mathrm{u}} \cdot 2 \mathrm{x}$
$\frac{\mathrm{dA}}{\mathrm{dx}}=\sqrt{\mathrm{x}^{2}} \cdot 2 \mathrm{x}$
$\frac{\mathrm{dA}}{\mathrm{dx}}=|\mathrm{x}| \cdot 2 \mathrm{x}$
This example highlights the difference between the variables $t$ and $x$. While $t$ cannot be negative, $x$ actually can be. The absolute value sign in the derivative ensures that you will get a negative derivative when you select a negative value for $x$. That makes sense because as $x$ increases from say -5 to -4 , the value of the integral decreases from $\int_{0}^{25} \sqrt{\mathrm{t}} \mathrm{dt}$ to $\int_{0}^{16} \sqrt{\mathrm{t}} \mathrm{dt}$.

Again, note that the derivative will be the same regardless of the lower bound of the integral. We would get the same answer for $A(x)=\int_{3}^{x^{2}} \sqrt{t} d t$. That lower bound is a constant so it doesn't change.

## IX. Integration Techniques

## Integrating $\mathbf{x}^{\mathbf{1}}$

$$
\int \frac{1}{x} d x=\ln |x|+C
$$

When we looked at the derivative of the function $y=\ln x$, we saw that the derivative seems to have two parts, while the original function has only positive $x$-values since negative numbers don't have logarithms.


To correct for the missing part, we used $\mathrm{y}=\ln |\mathrm{x}|$ and its derivative $\mathrm{y}=\frac{1}{\mathrm{x}}$ :


We saw that $y=\ln |x|$ does in fact have the derivative $y^{\prime}=\frac{1}{x}$. That means that when you need the antiderivative of $x^{-1}$, you should use $\ln |x|$ rather than just $\ln \mathrm{x}$.
$\int \frac{1}{x} d x=\ln |x|+C$

## U-Substitution inside the Integral

Substitute one part of an integral by $u$, so that the other part is similar to $d u$.

Be careful with definite integrals, because the limits of integration will change too!

The idea of finding an antiderivative is simple, but in practice it can be quite difficult. We have already seen that $u$-substitution is helpful in many situations, and this is no exception. Let's do
a really simple example to see how it works. Consider the integral $\int \cos (13 x) d x$. To solve this integral we must find the antiderivative of $\cos (13 x)$, which you may or may not immediately be able to come up with. However, you already know the antiderivative of $\cos x$, which is $\sin x+C$. It may be a good idea to replace $13 x$ with $u$.

If $u=13 x$, then $\frac{d u}{d x}=13$, so $d u=13 d x$, and $d x=\frac{1}{13} d u$.
Making the appropriate replacements turns $\int \cos (13 x) d x$ into $\int \frac{1}{13} \cos u d u$. This is equal to $\frac{1}{13} \int \cos u d u$, and it solves to $\frac{1}{13} \sin u+C$. Now replace $u$ with $13 x$ to get $\frac{1}{13} \sin (13 x)+C$.

As we said before, to find the integral we take the antiderivative. If we found the correct
antiderivative, then the derivative of $\frac{1}{13} \sin (13 x)+C$ should be $\cos (13 x)$, and it is. ©
Next, let's try solving an integral involving a product: $\int 2 x \mathrm{e}^{\mathrm{x}^{2}} \mathrm{dx}$. As you know, the derivative of $\mathrm{e}^{\mathrm{x}}$ is $\mathrm{e}^{\mathrm{x}}$. The confusing part here is $\mathrm{x}^{2}$, so let's get rid of that with a u -substitution:
$u=x^{2}$
$\frac{d u}{d x}=2 x$, so $d u=2 x d x$
du can replace $2 x d x$. We can rewrite the integral as $\int e^{u} d u$, which is just $e^{u}+C$. The answer is $e^{x^{2}}+C$, which in fact has the required derivative $e^{x^{2}} \cdot 2 x$, by the chain rule.

Alternatively, you can substitute for $\mathrm{e}^{\mathrm{x}^{2}}$ :
$\mathrm{u}=\mathrm{e}^{\mathrm{x}^{2}}$
$d u=2 x e^{x^{2}} d x$
Oddly enough, now we don't need $u$ in the new integral at all; we can just write it as $\int d u$.
$\int d u=u+C=e^{x^{2}}+C$

## Example

Solve $\int \sec ^{2}(\sin (3 x)) \cos (3 x) d x$
Although this looks really complicated, the problem was designed to involve derivatives so we can use a u-substitution. First, we want to substitute for $3 x$ :
$u=3 x$
$d u=3 d x$
$\frac{1}{3} d u=d x$
$\frac{1}{3} \int \sec ^{2}(\sin (u)) \cos (u) d u$
There, that already looks less complicated. Next, we can substitute for $\sin (u)$, because the derivative of that is already in the integral. Since $u$ has already been used, we have to turn to the next letter, v :
$v=\sin (u)$
$d v=\cos (u) d u$
$\frac{1}{3} \int \sec ^{2}(v) d v$
$\frac{1}{3} \tan v+C$
Once I get to this stage I'm sometimes so pleased with myself that I forget to finish the problem. Since $v$ is not part of the original question you have to get rid of it:
$\frac{1}{3} \tan (\sin (u))+C$
And $u$ as well:
$\frac{1}{3} \tan (\sin (3 x))+C$
Don't forget to check your work when you are done! Take the derivative of your answer to see that you get the same expression that you were integrating.

After doing many similar problems, people tend to see a pattern. There is an unfortunate tendency to express such patterns as complex-looking abstract expressions. The problem we just did follows a pattern, so we could make it look like this:

## Example

Solve $\int f^{\prime}(g(3 x)) g^{\prime}(3 x) d x$

My first thought here is "Huh, what?", but if you look closely you can see what this expression means. $3 x$ is still there, so you can substitute for it. You can also see that the second part, $g^{\prime}(3 x)$, must be the derivative of something else we can substitute for, $g(3 x)$.

In fact, we can even use a single substitution to be more efficient:
$u=g(3 x)$
$d u=g^{\prime}(3 x) \cdot 3 d x$
$\frac{1}{3} d u=g^{\prime}(3 x) d x$
$\frac{1}{3} \int f^{\prime}(u) d u$
$\frac{1}{3} f(u)+C$
$\frac{1}{3} f(g(3 x))+C$
Notice that you can still take the derivative to check your work:
$\frac{1}{3} f^{\prime}(g(3 x)) \cdot g^{\prime}(3 x) \cdot 3=f^{\prime}(g(3 x))\left(g^{\prime}(3 x)\right)$

## Example

Solve $\int_{1}^{2} \frac{\ln \mathrm{x}}{2 \mathrm{x}} \mathrm{dx}$
Let $\mathrm{u}=\ln \mathrm{x}$. Then $\frac{\mathrm{du}}{\mathrm{dx}}=\frac{1}{\mathrm{x}}$, and $\mathrm{du}=\frac{1}{\mathrm{x}} \mathrm{dx}$. What we really want to substitute for is $\frac{1}{2 \mathrm{x}} \mathrm{dx}$, so use $\frac{\mathrm{du}}{2}=\frac{1}{2 \mathrm{x}} \mathrm{dx}$. Now the integral looks like this:
$\int_{1}^{2} \frac{1}{2} \mathrm{udu}$
Oops, now that we are using $u$ to integrate, the limits of integration are no longer correct!
Previously, the integral was expressing the area under $y=\frac{\ln x}{2 x}$ from $x=1$ to $x=2$. Now we are integrating with respect to u , and $\mathrm{u}=\ln \mathrm{x}$. When $\mathrm{x}=1, \mathrm{u}=\ln 1$, and when $\mathrm{x}=2, \mathrm{u}=\ln 2$.
Integrate from $\ln 1$ to $\ln 2$, like this:
$\frac{1}{2} \int_{\ln 1}^{\ln 2} \mathrm{udu}$
That solves as $\frac{1}{4} u^{2}\left[\ln 22=\frac{1}{4}(\ln 2)^{2}-\frac{1}{4}(\ln 1)^{2}=\frac{1}{4}(\ln 2)^{2}-0 \approx 0.12\right.$.

It is really easy to forget to change the limits of integration. Because I have done that many times, I prefer to solve the indefinite integral instead, and then evaluate it:
$\int \frac{\ln x}{2 x} d x=\frac{1}{2} \int u d u=\frac{1}{4} u^{2}+C=\frac{1}{4}(\ln x)^{2}+C \quad$ [Be careful, it is also easy to forget to put $x$ back in!] The constant C just cancels out when you evaluate the integral:
$\left(\frac{1}{4}(\ln x)^{2}+C\right)\left[\begin{array}{l}2 \\ 1\end{array}=\frac{1}{4}(\ln 2)^{2}+C-\left(\frac{1}{4}(\ln 1)^{2}+C\right)=\frac{1}{4}(\ln 2)^{2}+C-0-C \approx 0.12\right.$.

Most u-substitution problems can be solved with some trial and error, provided you know that this is the technique you are expected to use. Once you learn more integration techniques things can get a bit harder.

## SOME EXAMPLES OF INTEGRALS THAT CAN BE SOLVED BY U-SUBSTITUTION

1. $\int \frac{\ln x}{x} d x$
2. $\int \frac{1}{x \sqrt{1-\ln ^{2} x}} d x$
3. $\int \frac{x}{\sqrt{1-2 x^{2}}} d x$
4. $\int x^{5} \sqrt{1+x^{2}} d x$
5. $\int \frac{\cos x}{\left(1+\sin ^{2} x\right)} d x$
6. $\int \frac{\mathrm{e}^{\frac{1}{x}}}{\mathrm{x}^{2}} \mathrm{dx}$
7. $\int(1+\sqrt{\mathrm{x}})^{6} d x$
8. $\int \frac{x^{2}}{9+x^{6}} d x$

## Solutions

1. $u=\ln x, d u=\frac{1}{x} d x . \rightarrow \int u d u \rightarrow \frac{1}{2} u^{2}+C \rightarrow \frac{1}{2}(\ln x)^{2}+C$
2. $u=\ln x, d u=1 / x d x . \rightarrow \int \frac{1}{\sqrt{1-u^{2}}} d u \rightarrow \sin ^{-1} u+C \rightarrow \sin ^{-1}(\ln x)+C$ (see "Inverse Trigonometric Functions")
3. $u=1-2 x^{2}, d u=-4 d x, d x=-\frac{1}{4} d u . \rightarrow \int \frac{-\frac{1}{4}}{\sqrt{u}} d u \rightarrow-\frac{1}{4} \int \frac{1}{\sqrt{u}} d u \rightarrow-\frac{1}{4} \int u^{-1 / 2} d u \rightarrow$ $-\frac{1}{2} u^{1 / 2}+C \rightarrow-\frac{1}{2}\left(1-2 x^{2}\right)^{1 / 2}+C$
4. $u=1+x^{2}, d u=2 x d x, x d x=\frac{1}{2} d u, x^{4}=(u-1)^{2}$. $\rightarrow \frac{1}{2} \int(u-1)^{2} u^{1 / 2} d u \rightarrow$ $\frac{1}{2} \int\left(u^{5 / 2}-2 u^{3 / 2}+u^{1 / 2}\right) d u \rightarrow \frac{1}{7}\left(1+x^{2}\right)^{7 / 2}-\frac{2}{5}\left(1+x^{2}\right)^{5 / 2}+\frac{1}{3}\left(1+x^{2}\right)^{3 / 2}+C$
5. $u=\sin x, d u=\cos x d x \rightarrow \int \frac{1}{1+u^{2}} d u=\tan ^{-1} u+C \rightarrow \tan ^{-1}(\sin x)+C$
6. $u=1 / x=x^{-1}, d u=-x^{-2} d x,-d u=\frac{1}{x^{2}} d x \rightarrow-\int e^{u} d u \rightarrow-e^{u}+C \rightarrow-e^{1 / x}+C$
7. $u=1+\sqrt{\mathrm{x}}, \mathrm{du}=\frac{1}{2} \mathrm{x}^{-1 / 2} \mathrm{dx}, \mathrm{du}=\frac{1}{2 \sqrt{\mathrm{x}}} \mathrm{dx}$. Express this in terms of u , which is possible because $\sqrt{x}=u-1: d u=\frac{1}{2(u-1)} d x$. Now $d x=2(u-1) d u \rightarrow 2 \int u^{6}(u-1) d u \rightarrow 2 \int\left(u^{7}-u^{6}\right) d u \rightarrow \frac{1}{4}(1+$ $\sqrt{\mathrm{x}})^{8}-\frac{2}{7}(1+\sqrt{\mathrm{x}})^{7}+C$
8. $\int \frac{x^{2}}{9\left(1+\frac{x^{6}}{9}\right)} d x \rightarrow \frac{1}{9} \int \frac{x^{2}}{1+\left(\frac{x^{3}}{3}\right)^{2}} d x . u=\frac{x^{3}}{3}, d u=x^{2} \rightarrow \frac{1}{9} \tan ^{-1}\left(\frac{x^{3}}{3}\right)+C$

## Tricks for Solving Integrals

1. Awkward Fractions. Suppose that we are trying to solve an integral like $\int \frac{x}{x+a} d x$, where $a$ is some constant. The first thing you may think of is to do a $u$-substitution with $u=x+a$, and du $=d x$. But what about the $x$ on the top? Well, it is actually easy to get rid of $x$ by expressing it in terms of $u$ : $x=u-a$.
$\int \frac{\mathrm{u}-\mathrm{a}}{\mathrm{u}} \mathrm{d} u=\int\left(\frac{\mathrm{u}}{\mathrm{u}}-\frac{\mathrm{a}}{\mathrm{u}}\right) \mathrm{du}=\int\left(1-\frac{\mathrm{a}}{\mathrm{u}}\right) \mathrm{du}=\mathrm{u}+\ln |\mathrm{u}|+C$.

Since $u=x+a$, we get $x+a-a \ln |x+a|+C$. Remember that $a$ is just some constant. Rewrite the expression as $x-a \ln |x+a|+a+C$, and combine $a$ and $C$ into a single constant. The final solution is $x-a \ln |x+a|+C$.

Another option is to change the fraction using long division. $x \div(x+a)$ works out to $1-\frac{a}{x+a}$, which is easy to integrate:
 The result is 1 with a remainder of $-a$, so we get $1+\frac{-a}{x+a}$ : $\int\left(1-\frac{a}{x+a}\right) d x$. This integral is much easier to handle than the original, solving to produce $x-a \ln |x+a|+C$ as before .
2. Division. If you have a polynomial fraction with greatest power on top larger than on the bottom:
$\int \frac{x^{2}+3 x-7}{x-2} d x$ : Do long division to get $\int\left(x+5+\frac{3}{x-2}\right) d x$.
3. Square Roots. It is possible to get rid of a square root by making a u-substitution for it and then squaring $u$. For example, while the integral $\int x \sqrt{x+1} d x$ could be solved by a $u$ substitution for $x+1$, we can also substitute $u=\sqrt{x+1}$. Now square $u$ to get $u^{2}=x+1$, so that $x=u^{2}-1$. Just be a bit careful when you differentiate $u^{2}=x+1$, because you have to use implicit differentiation with respect to $x: 2 u d u=d x$. Then rewrite the integral as $\int\left(u^{2}-1\right) \cdot u \cdot$ $2 u d u$. Then you can just simplify and solve. $2 \int\left(u^{4}-u^{2}\right) d u=\frac{2}{5} u^{5}-\frac{2}{3} u^{3}+C$.

Don't forget that it is $\mathrm{u}^{2}$ that is equal to $\mathrm{x}+1$ rather than u . Rewrite the expression as $\frac{2}{5}\left(u^{2}\right)^{\frac{5}{2}}-\frac{2}{3}\left(u^{2}\right)^{\frac{3}{2}}+C=\frac{2}{5}(x+1)^{\frac{5}{2}}-\frac{2}{3}(x+1)^{\frac{3}{2}}+C$. If it makes your teacher happy you could factor out ( $x+1$ ) and maybe create a common denominator for those fractions.
4. Integrals of Exponential Functions. Because we know that the derivative of $\mathrm{e}^{\mathrm{x}}$ is just $\mathrm{e}^{\mathrm{x}}$, we can say that $\int e^{x} d x=e^{x}+C$. To find $\int e^{5 x} d x$, just consider that the derivative of $e^{5 x}$ would be $5 e^{5 x}$. We need to divide by 5 to get the integral: $\int e^{5 x} d x=\frac{1}{5} e^{5 x}+C$. Now things work out if you take the derivative.

So how about $\int 4^{x} d x$ ? To get a handle on this exponential function we have to rewrite it using $e$ as the base. $4=e^{\ln 4}$, so the integral becomes $\int\left(e^{\ln 4}\right)^{x}=\int e^{\ln 4 \cdot x} d x$. The logarithm makes
that look a bit scary, but it is really no different than $\int e^{5 x} d x . \int e^{x \ln 4} d x=\frac{1}{\ln 4} e^{x \ln 4}+C$, which you can then change back to $\frac{1}{\ln 4} 4^{x}+C$.
5. Factor. Whenever you are stuck, ask yourself: "Can it be factored?" For example,
$\int \sqrt{\frac{4 x^{2}+4 x+1}{4 x}} d x=\int \sqrt{\frac{(2 x+1)^{2}}{4 x}} d x=\int \frac{2 x+1}{2 \sqrt{x}} d x=\int\left(x^{1 / 2}+\frac{1}{2} x^{-1 / 2}\right) d x$
6. Complete the Square. $x^{2}-10 x+26=x^{2}-10 x+25-25+26=(x-5)^{2}+1$. Sometimes this will allow you to simplify an expression and then integrate it.
7. Multiply to take advantage of the difference of two squares. For example, $\int \sqrt{1+\sin x} d x$ can be solved by multiplying by $\frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}$. This gives $\int \frac{\sqrt{1-\sin ^{2} x}}{\sqrt{1-\sin x}} d x=\int \frac{\sqrt{\cos ^{2} x}}{\sqrt{1-\sin x}} d x$ $=\int \frac{\cos x}{\sqrt{1-\sin x}} d x$, which allows you to substitute $u=1-\sin x$. Caution: $\sqrt{\cos ^{2} x}$ is only equal to $\cos$ $x$ for positive values of the cosine. $y=\frac{\cos x}{\sqrt{1-\sin x}}$ is not the same function as $y=\sqrt{1+\sin x}$. It does not exist at $\frac{\pi}{2}$, and it becomes negative right after that. If you need $\int_{0}^{\pi} \sqrt{1+\sin x} d x$, use $\int_{0}^{\pi}\left|\frac{\cos x}{\sqrt{1-\sin x}}\right| \mathrm{dx}$.

## Trigonometric Integrals

$$
\begin{array}{ll}
\int \sin x d x=-\cos x+C & \int \cos x d x=\sin x+C \\
\int \sec ^{2} x=\tan x+C & \int-\csc ^{2} x d x=\cot x+C \\
\int \sec x \tan x d x=\sec x+C & \int-\csc x \cot x d x=\csc x+C
\end{array}
$$

## Function

## Derivative

$f(x)=\sin ^{-1} x$

$$
\frac{1}{\sqrt{1-x^{2}}}
$$

$$
\rightarrow \int \frac{1}{\sqrt{1-x^{2}}} d x=\sin ^{-1} x+C
$$

$f(x)=\cos ^{-1} x$
$\frac{-1}{\sqrt{1-x^{2}}}$
$\rightarrow \int \frac{-1}{\sqrt{1-x^{2}}} d x=\cos ^{-1} x+C$
$f(x)=\tan ^{-1} x \quad \frac{1}{1+x^{2}} \quad \rightarrow \int \frac{1}{1+x^{2}} d x=\tan ^{-1} x+C$

$$
\begin{array}{ll}
\int \tan x d x=\ln |\sec x|+C & \int \cot x d x=\ln |\sin x|+C \\
\int \sec x d x=\ln |\sec x+\tan x|+C & \int \csc x d x=\ln |\csc x-\cot x|+C
\end{array}
$$

If you need the integral of the square of the sine or cosine function, use the double angle formulas: $\sin ^{2} x=\frac{1-\cos (2 x)}{2}$ and $\cos ^{2} x=\frac{1+\cos (2 x)}{2}$.

Attempts to climb Mount Everest have resulted in many deaths and serious injuries, but people just have to get try to get to the top "because it's there." Trigonometric functions are there too, so people feel compelled to integrate them, as well as every single one of their various combinations and variations. Fortunately, you already know a few:
$\int \sin \mathrm{xdx}=-\cos \mathrm{x}+\mathrm{C}$
$\int \cos x d x=\sin x+C$
$\int \sec ^{2} \mathrm{x}=\tan \mathrm{x}+\mathrm{C}$
$\int-\csc ^{2} x d x=\cot x+C$
$\int \sec x \tan x d x=\sec x+C$
$\int-\csc x \cot x d x=\csc x+C$
$\int \tan \mathrm{x} d \mathrm{~d}=\int \frac{\sin \mathrm{x}}{\cos \mathrm{x}} \mathrm{dx}=$ ?
To find the integral of the tangent function, we can use a $u$-substitution. Let $u=\cos x$, so $d u=$ $-\sin x d x$. The integral becomes $\int \frac{1}{\cos x} \sin x d x=-\int \frac{1}{u} d u$, which solves as $-\ln |u|+C$. Put the cosine back where it was: $-\ln |\cos x|+C$. Now you can use a clever trick, based on the fact that $\ln 1$ is zero, and that $\ln \mathrm{a}-\ln \mathrm{b}=\ln \frac{\mathrm{a}}{\mathrm{b}}$ :
$-\ln |\cos x|=0-\ln |\cos x|=\ln 1-\ln |\cos x|=\ln \left|\frac{1}{\cos x}\right|=\ln |\sec x|$.
$\int \tan \mathrm{xdx}=\int \frac{\sin \mathrm{x}}{\cos \mathrm{x}} \mathrm{dx}=\ln |\sec \mathrm{x}|+C$

## Practice

Show how to find the integral of the cotangent:
$\int \cot \mathrm{xdx}=\ln |\sin \mathrm{x}|+C$
$\int \sec x d x=?$
We can only speculate what injuries may have resulted from attempts to solve this integral, but eventually someone was successful. It is not exactly obvious, but if we multiply by $\frac{\sec x+\tan x}{\sec x+\tan x}$ we can use $u$-substitution. The integral turns into $\int \frac{\sec ^{2} x+\sec x \tan x}{\sec x+\tan x} d x$, and now you can use $u$ $=\sec x+\tan x$. Then $d u=\left(\sec x \tan x+\sec ^{2} x\right) d x$. The new integral $\int \frac{1}{u} d u$ solves as $\ln |u|+C=$ In $|\sec x+\tan x|+C$. If you try this same type of trick for the integral of the cosecant, you will find that it is most convenient to multiply by a factor of $\frac{\csc x-\cot x}{\csc x-\cot x}$. Substitute $u=\csc x-\cot x$ to solve the integral. So,
$\int \sec x d x=\ln |\sec x+\tan x|+C$
$\int \csc x d x=\ln |\csc x-\cot x|+C$

If you need the integral of the square of the sine or cosine function, use the double angle formulas: $\sin ^{2} x=\frac{1-\cos (2 x)}{2}$ and $\cos ^{2} x=\frac{1+\cos (2 x)}{2}$.
$\int \sin ^{2} \mathrm{xdx}=\int \frac{1-\cos (2 \mathrm{x})}{2} \mathrm{dx}=\int \frac{1}{2} \mathrm{dx}-\int \frac{\cos (2 \mathrm{x})}{2} \mathrm{dx}=\frac{1}{2} \mathrm{x}-\frac{1}{4} \sin (2 \mathrm{x})+\mathrm{C}$
Since $\sin (2 x)=2 \sin x \cos x$, this solution can also be written as $\frac{1}{2} x-\frac{1}{2} \sin x \cos x+C$
$\int \cos ^{2} x d x=\int \frac{1+\cos (2 x)}{2} d x=\int \frac{1}{2} d x+\int \frac{\cos (2 x)}{2} d x=\frac{1}{2} x+\frac{1}{4} \sin (2 x)+C$
This same strategy works for $\int \sin ^{4} x d x$, since you can write that as $\int\left(\sin ^{2} x\right)^{2} d x$. Then you can change that to $\int\left(\frac{1-\cos (2 x)}{2}\right)^{2} d x$, which is $\int \frac{1-2 \cos (2 x)+\cos ^{2}(2 x)}{4} d x$. That contains another square, so rinse and repeat:
$\int \frac{1}{4} d x-\int \frac{2 \cos (2 x)}{4} d x+\int \frac{\cos ^{2}(2 x)}{4} d x=\frac{1}{4} x+\frac{1}{2} \int \cos (2 x) d x+\frac{1}{4} \int \frac{1+\cos (4 x)}{2} d x$
That works out to $\frac{1}{4} x-\frac{1}{4} \sin (2 x)+\frac{1}{8} x+\frac{1}{32} \sin (4 x)+C,=\frac{3}{8} x-\frac{1}{4} \sin (2 x)+\frac{1}{32} \sin (4 x)+C$.

## Practice

Show that $\int \cos ^{4} x d x=\frac{3}{8} x+\frac{1}{4} \sin (2 x)+\frac{1}{32} \sin (4 x)+C$.

Now let's look at integrals containing both sine and cosine. For something like $\int \sin ^{3} \mathrm{x} \cos \mathrm{xdx}$, you can just use a $u$-substitution. If $u=\sin x$, then $d u=\cos x d x$. Solve $\int u^{3} d u$, and you'll end up with $\frac{1}{4} \sin x+C$. Unfortunately that won't work so well for $\int \sin ^{3} x \cos ^{2} x d x$. Write this integral as $\int \sin ^{2} x \cos ^{2} x \sin x d x$ so you can use the identity $\sin ^{2} x+\cos ^{2} x=1$. Since $\sin ^{2} x=$ $1-\cos ^{2}$ the integral turns into $\int\left(1-\cos ^{2} x\right) \cos ^{2} x \sin x d x$. That is the same as $\int\left(\cos ^{2} x-\cos ^{4} x\right) \sin x d x$. Split it into two integrals: $\int \cos ^{2} x \sin x d x-\int \cos ^{4} x \sin x d x$. Here a $u$-substitution works well, with $u=\cos x$ and $d u=-\sin x$. The final result is $\frac{1}{5} \cos ^{5} x-\frac{1}{3} \cos ^{3} x+C$.

When you check your answer here, you may be surprised that the derivative of this expression is $-\cos ^{4} x \sin x+\cos ^{2} x \sin x$, rather than $\sin ^{3} x \cos ^{2} x$ as you were expecting. You have to rearrange things a bit: $-\cos ^{4} x \sin x+\cos ^{2} x \sin x=\cos ^{2} x \sin x\left(1-\cos ^{2} x\right)=\cos ^{2} x \sin x\left(\sin ^{2} x\right)=\sin ^{3} x \cos ^{2} x$.

The same tricks we just used will also work for the tangent and the secant, which are related by the identity $\tan ^{2} x+1=\sec ^{2} x$.

There are many, many other trigonometric integrals. Wikipedia has a long list of them, neatly organized by type.

## X. Using Integrals

## The Average Value of a Function

The average value of a function on the interval from $a$ to $b$ is $\frac{\int_{a}^{b} f(x) d x}{b-a}$

Calculus can be used to determine the average value of a function over a certain interval. To help us make sense of what this means, we will use speed as an example. If you look at the illustration below, you will see that the speed of some object is increasing from 0 miles per second to 5 miles per second over a period of 4 seconds.


What is the average speed of this object for this time interval? We don't need calculus here, because there is a nice steady increase in the speed. At the beginning of the interval the speed was 0 , and at the end the speed is 5 miles per second. The average speed was 2.5 miles per second. Notice that at a steady speed of 2.5 miles per second for 4 seconds the object would travel a total distance of 10 miles. Distance is equal to speed multiplied by time. The distance traveled is also equal to the area underneath the speed curve. The area under this speed curve is exactly equal to 10 . So if we know that the distance traveled is 10 miles, over a period of 4 seconds, we can figure out that the average speed has to be 2.5. The illustration below shows that the distance traveled would be the same if the object proceeded at a speed of 2.5 miles per second during the entire interval:


Average speed $=$ Total distance traveled $\div$ time .

Now look at this picture:


Hmm, how can we find the average speed now? Well, things still work the same way.
Average speed $=$ Total distance traveled $\div$ time.
The total distance traveled is the area under the curve, which we can find by taking the integral. Chop up the time into infinitely many infinitely tiny intervals dt. Each interval is so small that the speed really doesn't change from the start of the interval to the end, so we can use the height of the function at some point during that time. For one little interval, the distance traveled is $f(t) \cdot d t$. Now add up all those little distances: $\int_{0}^{4} f(t) d t$ is the total distance traveled between 0 and 4 seconds, and in this case that is $\int_{0}^{4} 3 \sqrt{\mathrm{t}} \mathrm{dt}$.
$\int_{0}^{4} 3 \sqrt{\mathrm{t}} \mathrm{dt}=3 \int_{0}^{4} \mathrm{t}^{\frac{1}{2}} \mathrm{dt}=3 \cdot \frac{2}{3} \mathrm{t}^{\frac{3}{2}}\left[\begin{array}{l}4 \\ 0\end{array}=2 \mathrm{t}^{\frac{3}{2}}\left[\begin{array}{l}4 \\ 0\end{array}=2 \sqrt{4}^{3}-0=2 \cdot 8=16\right.\right.$.
The average speed is distance $\div$ time, which is $\frac{16}{4}=4$. The picture below shows that the distance traveled would also be 16 if the speed had been 4 the entire time:


This method of finding the average speed works for any kind of function.
We can find the average value of the function on some given interval by taking the area under the curve and dividing it by the length of the interval.

## The Area between Two Curves

The area between two curves is $\int(f(x)-g(x)) d x$, if $f(x) \geq g(x)$
To find the limits of integration, set $\mathrm{f}(\mathrm{x})$ equal to $\mathrm{g}(\mathrm{x})$ to get the intersect points.

Most students find this subject rather intuitive. Just like we did before, we can find the area between the graphs of two functions by dividing it up into infinitely many infinitely thin strips. Each strip has a width of $d x$, and we can find the height of a sample strip by taking the $y$-value of the top curve and subtracting the $y$-value of the bottom curve. If $f(x)$ has a larger value that $g(x)$, the infinitely tiny area $d A$ of each strip is $(f(x)-g(x)) d x$. The total area $A$ is $\int d A$, which is equal to $\int(f(x)-g(x)) d x$. That is really no different from finding the area between $f(x)$ and the $x$-axis $(y=0)$, which is given by the integral $\int(f(x)-0) d x$.

Consider the area between $y=x$ and $y=x^{2}$. If no limits are given for the integral, you must find the intersect points of the two curves. To do that, just set them equal to each other and solve as you learned to do in algebra. $x=x^{2}$, so $x-x^{2}=0$ which says that $x(x-1)=0$. The two intersect points are $x=0$ and $x=1$. Draw the graphs of the two functions to see that over this interval $y=x$ has larger values than $y=x^{2}$. The area between the two graphs is given by the integral $\int_{0}^{1}\left(x-x^{2}\right) d x$.

If you make a mistake and use the function with the larger $y$-value as the second function $g(x)$ in the integral $\int(f(x)-g(x)) d x$, you will obtain a negative area, which should then tell you that you did it the wrong way around. However, this is not a substitute for drawing a picture of the situation.

## Example 1

Find the area between the graphs of the functions $y=\sin (x)+2$ and $y=2$, between the $y$ axis and the line $\mathrm{x}=2 \pi$.

If you fail to draw a picture in this case, you might try to find the integral $\int_{0}^{2 \pi}(\sin (\mathrm{x})+2-2) \mathrm{dx}$, which will give you an area of zero. The graph of these functions
shows that you need to take the integral in two parts:


From 0 to $\pi, y=\sin (x)+2$ is the top function. The thin blue line represents one of the infinitely tiny pieces that we are integrating. The height of this line is $(\sin (x)+2)-2$, and its width is $d x$. The integral is $\int_{0}^{\pi}(\sin (x)+2)-2 d x=\int_{0}^{\pi} \sin (x) d x=-\cos x\left[\begin{array}{l}\pi \\ 0\end{array}=-1--1=2\right.$.

From $\pi$ to $2 \pi, y=2$ is the top function. The integral is $\int_{\pi}^{2 \pi} 2-(\sin (x)+2) d x$. That simplifies to $\int_{\pi}^{2 \pi}-\sin (x)$, which is $\cos x\left[\begin{array}{c}2 \pi \\ \pi\end{array}=1--1=2\right.$.

The total area between the graphs is 4 square units.

You may encounter problems where the graphs are described with $x$ as a function of $y$, such as $x=y^{2}$. Again, it is important to draw a picture. If you are used to using your graphing calculator, you may wonder what to do in this situation. You actually have two options here. The TI-84 graphing calculator can draw the inverse of the function $y=x^{2}$ for you, so that you then have a picture of $x=y^{2}$. To do this, enter the function $y=x^{2}$ into the first available function slot, $Y_{1}$. You don't actually want to see this function, so we need to keep it from showing in the final plot. Select the small slanted line just in front of the $Y$, and press enter until the line changes to a little open circle. Now you can leave this screen (by pressing $2^{\text {ND }}$ MODE). Next, use the DRAW function, found above the PRGM button. Option 8 in the DRAW menu is Drawlnv. To enter $Y_{1}$, you need to press the VARS key and select $Y$-VARS. Then select Function, and you will be able to choose $Y_{1}$ from the list that appears. Once you have done this, your calculator will display DrawInv $Y_{1}$, but it will not actually draw the graph until you press ENTER.

If you can't remember the proper calculator commands and you really need to see a graph of $x$ as a function of $y$, you can also use my revolutionary new non-patented Axis Inverter. Its unique mechanical design works with any model graphing calculator. A $\$ 0.01$ value, it is available at absolutely no charge to readers of this e-book. Act now, and you'll also get the fun of cutting it out and gluing the front and back parts together in the orientation shown.



The Axis Inverter reflects the x and y axes over the line $\mathrm{y}=\mathrm{x}$, providing the proper orientation for the graph of the inverse function without having to reflect the graph itself. It fits on top of the axes in your calculator window, so that you can just enter $y=x^{2}$ and it will display as $x=y^{2}$. Note that after entering the functions you integrate using dy rather than dx , because the width of the infinitely tiny strips you are using will lie along the $y$-axis.

## Example 2

Find the area between $\mathrm{x}=\mathrm{y}^{2}$ and $\mathrm{x}=-\mathrm{y}^{2}+8$.

Just draw these functions as $\mathrm{y}=\mathrm{x}^{2}$ and $\mathrm{y}=-\mathrm{x}^{2}+8$. In the image below, I have used the axis
inverter to show the inverse functions we need. The on-indicator is on, and it is reminding me to use $d y$ instead of $d x$ :


The top function is $x=-y^{2}+8$, and the bottom function is $x=y^{2}$. Set the two functions equal to each other to see where they intersect:
$-y^{2}+8=y^{2}$
$8=2 y^{2}$
$4=y^{2}$
$y= \pm \sqrt{4}$
The curves intersect at $y=-2$ and $y=2$. We will divide the area between these $y$-values up into infinitely many infinitely small strips with a width of $d y$. The height of each strip is the top $x$ value, $-y^{2}+8$, minus the bottom $x$-value, $y^{2}$. $d A=\left(-y^{2}+8-y^{2}\right) d y$. Now integrate:
$\int d A=\int_{-2}^{2}\left(-2 y^{2}+8\right) d y$
$A=-\frac{2}{3} y^{3}+8 y\left[\begin{array}{c}2 \\ -2\end{array}=-\frac{2}{3} \cdot 2^{3}+8 \cdot 2-\left(-\frac{2}{3} \cdot(-2)^{3}+8(-2)\right)=21 \frac{1}{3}\right.$

Interestingly, we can also use integrals to find the area between two circles with the same center. Suppose the outer circle has a radius of 5 cm , and the inner circle has a radius of 3 cm . It is easy to find the area between these circles without using calculus: $25 \pi-9 \pi=16 \pi$. To use calculus, we should look back at the topic "how a circle grows". If you think about how the area increases, you can see that $d A=2 \pi r d r$. If we want the area from $r=3$ to $r=5$, we can take the integral: $A=\int_{3}^{5} 2 \pi r d r=\pi r^{2}\left[\begin{array}{l}5 \\ 3\end{array}=16 \pi\right.$.

This kind of thing becomes more important when there is an additional dimension to the problem.

## Example 2



Photo: Tim Vickers
The population density of water striders on the surface of a circular pond is modeled by the equation $d=0.015 r^{2}$ (density in insects per square yard of surface area, $r=o$ at the center of the pond). The radius of the pond is 25 yards. Find the approximate number of water striders along the outer edge of the pond, within 2 yards of the shore.

Here the number of bugs per square yard depends on the distance from the center of the circle. If you consider a really thin ring 2 yards from the shore, the density of the bugs is approximately the same everywhere along that ring. We can divide the entire surface area of the pond up into
infinitely thin rings, and multiply the area of each ring by the density of insects at that particular point. For a sample ring the area is $2 \pi r d r$, just as we saw earlier. The number of insects in this area is given by $2 \pi r d r \cdot 0.015 r^{2}$. Now add up all the areas from $r=23$ to $r=25$ :
$\int_{23}^{25}(2 \pi r)\left(.015 r^{2}\right) d r=\int_{23}^{25} 0.03 \pi r^{3} d r=0.03 \pi \int_{23}^{25} r^{3} d r$.
That works out to $0.03 \pi\left(\frac{1}{4} r^{4}\left[\begin{array}{l}25 \\ 23\end{array}\right) \approx 2610\right.$ water striders.
Oddly enough, this works even though it doesn't actually make any sense to consider the density of insects in an infinitely thin ring.

## Practice

Use calculus to find the area between $f(x)=|x-4|+2$ and the line $y=6$. Check your answer using geometry.

## Using Integrals to Find Volume

## 1. Slices

$$
\begin{aligned}
& \text { Round slices: } V=\int \pi r^{2} d h \\
& \text { Square slices: } V=\int s^{2} d h
\end{aligned}
$$

The idea behind using integrals to find the volume of something is actually rather simple. Just grab an imaginary cereal box and use calculus to determine its volume. We will cut the box into an infinite number of infinitely thin slices, and then add up the volume of all those little slices to find the volume of the box. A single infinitely thin slice represents dV , an infinitely small change in the volume $V$. The total volume is the sum of all of the infinitely thin slices that we cut the box up into: $V=\int d V$.

Our imaginary cereal box has the convenient dimensions of 2 inches for the width, 8 inches for the length, and 12 inches for the height. We could slice the box up in any one of three different
directions, but let's just make our slices from top to bottom. That means that each slice will have a surface area of 2 times 8 or 16 square inches. That surface is actually a very thin rectangle of cardboard with a lot of infinitely thin bits of cereal in the middle. Most calculus problems use $x$ for the variable, but here we will use $h$ for the height of the box. Each slice has an infinitely small thickness that we will indicate by dh . The volume of one slice, dV , is 16 square inches times $d$ inches, or 16 dh cubic inches: $d V=16 \mathrm{dh}$. Once you have determined this, you have already solved your problem. Notice that $\frac{d V}{d h}=16$, that is, the derivative of the Volume (with respect to the height of the box), is 16. To get the Volume function, we need the antiderivative. If $\mathrm{V}=16 \mathrm{~h}$, then $\frac{\mathrm{dV}}{\mathrm{dh}}$ would be 16 . In fact, any function that looks like $V=16 \mathrm{~h}+$ some constant C would do the job.

Now that we know what we are looking for, let's use some integral notation:
$V$ is equal to the sum of all the infinitely tiny $d V$ 's that we have divided it into: $V=\int d V$. Since $\mathrm{dV}=16 \mathrm{dh}$, we take the integral on both sides:
$\int d V=\int 16 d h$
$V=\int 16 d h$
$V=16 h+C$
Now all we have to do is find the volume when $h$ is 12 inches; that is, we want the volume from $h=0$ to $h=12$ :
$V(12)=\int_{0}^{12} 16 \mathrm{dh}$
That works out to $16 \mathrm{~h}\left[\begin{array}{c}12 \\ 0\end{array}\right.$ (the Volume function evaluated at 12 minus the Volume function evaluated at 0 ), which is $16 \cdot 12-16 \cdot 0=192$ cubic inches. This is the same volume you would get if you multiply the length times the width times the height of the box, but then you don't get the fun of cutting up a box filled with cereal into infinitely many slices.

## Example 1

Find the volume of a 10 cm tall cylinder with a radius of 3 cm .
Although we could put this cylinder into an $x-y$ coordinate system, it is simpler to just slice it up from top to bottom using the variable $h$ for the height. Each infinitely thin slice has a volume of
$\pi r^{2} d h$, which is the base times the height. Because the radius is always 3 that would be $9 \pi d h$. $\mathrm{dV}=9 \pi \mathrm{dh}$. Now take the integral on both sides:
$\int d V=\int 9 \pi d h$
integrate from $h=0$ to $h=10$, remembering that $\pi$ is just a number:
$9 \pi \int_{0}^{10} \mathrm{dh}=9 \pi \mathrm{~h}\left[\begin{array}{c}10 \\ 0\end{array}=90 \pi \mathrm{~cm}^{3}\right.$

Of course calculus books can't have you do simple problems like this because you might get the impression that calculus is easy, or worse, unnecessary. Let's tackle a problem that would be impossible to solve if you had forgotten the formula for the volume of a cone. (Where does that formula come from anyway?)

## Example 2

Find the volume of the cone that would be generated by taking the line $\mathrm{y}=\mathrm{x}$, for $\mathrm{o} \leq \mathrm{x} \leq 1$, and rotating it around the $y$-axis.

This problem can be solved in the same way as we found the volume of the cereal box. We simply take slices of the cone, from the bottom to the top. Each slice will have the shape of a circle, and be infinitely thin. Because we are cutting along the $y$-axis, the thickness of each slice will be $d y$. The radius for every slice will be $x$, which would make the area $\pi x^{2}$. Since the thickness is $d y$, the volume of each slice is $\pi x^{2} d y$. Now we add up all of our slices from the bottom to the top along the $y$-axis. Since we are considering $x$ values between 0 and 1 , the top of the cone will be at $y=1$. The integral is $\int_{0}^{1} \pi x^{2} d y$. When you end up with an integral like this, which no you can't solve, it is important to realize that you have done nothing wrong. The integral is constructed correctly; it just needs a change of variables. Since we have a simple equation, $y=x$, we can easily change our integral to $\int_{0}^{1} \pi y^{2} d y$.
$\pi \int_{0}^{1} y^{2} d y=\pi \frac{y^{3}}{3}\left[\begin{array}{l}1 \\ 0\end{array}=\frac{1}{3} \pi\right.$.
Compare this answer to what you would get from the formula for the volume of a cone: 1/3 base times height.

## 2. Washers

For washers, $V=\int\left(\pi\left(r_{1}\right)^{2}-\pi\left(r_{2}\right)^{2}\right) d h$, where $r_{2}$ is the radius of the hole Rearrange to express $x$ in terms of $y$ as needed.

## Example

Consider the area inside a circle with radius 1 centered at the origin, bounded by the line $y=x$ and the $x$-axis. Find the volume of the solid generated by rotating this area around the $y$-axis.

I'm not good at drawing anything, but experience has taught me that it is important to create a cross-section picture for these rotation problems so you don't make mistakes. I highly recommend it.


The colored area represents half of the cross-section of the shape described. We will solve this problem by slicing up the shape from the bottom to the top, just as we did with the cereal box
earlier. Because of the direction in which we take the slices, each slice will have a thickness of dy. The infinitely thin slices will have a hole in the middle due to the slanted line at the top (except for the bottom slice). The slices will be perfectly round, with a perfectly round hole in the middle. Such shapes are called washers. The surface area of each washer is the total surface area of the circle minus the surface area that is missing due to the hole.

First we need to find the radius of each washer, which is the distance from the $y$-axis to the edge of the circle. Whenever you are measuring in a horizontal direction, think in terms of $x$. At any given height $y$, the radius of the corresponding washer is $x$. The equation of the circle is $y^{2}+x^{2}=1$, so $x=\sqrt{1-y^{2}}$. The total surface area of the washer without the hole is $\pi r^{2}$, which is $\pi x^{2}$. Because we are using $d y$, we need to express everything in terms of $y$. For this function $x^{2}=1-y^{2}$, so the surface area is $\pi\left(1-y^{2}\right)$. The area of the hole must be subtracted from this. If you look at your picture you will see that the radius of each hole is determined by our second function $y=x$. At each point $y$, the radius of the hole is $x$, which is equal to $y$. The area of the hole is $\pi y^{2}$. The net area of each washer is $\pi\left(1-y^{2}\right)-\pi y^{2}$. That rearranges to $\pi-\pi y^{2}-\pi y^{2}$, or $\pi-2 \pi y^{2}$. Each washer has a thickness of $d y$, so the volume of a sample washer is $\left(\pi-2 \pi y^{2}\right) d y$.

We have to find just how tall our shape is so we can add up the washers. As I vaguely recall my days in a trigonometry classroom, I see that since the line $y=x$ runs at a 45 degree angle, the height of the shape should be the sine of 45 degrees. However, if that doesn't stand out for you, you can figure out where the line and the circle intersect:
$x=y \quad$ and $\quad x=\sqrt{1-y^{2}}$
They intersect when $y=\sqrt{1-y^{2}}$ :
$y^{2}=1-y^{2}$
$2 y^{2}=1$
$\mathrm{y}^{2}=\frac{1}{2}$, so the positive value for y is $\frac{1}{\sqrt{2}}$ which is more properly written as $\mathrm{y}=\frac{\sqrt{2}}{2}$ (just to keep math teachers happy).

Now we integrate from 0 to $\frac{\sqrt{2}}{2}$ :
$\int_{0}^{\frac{\sqrt{2}}{2}}\left(\pi-2 \pi y^{2}\right) d y$
$\int_{0}^{\frac{\sqrt{2}}{2}} \pi d y-2 \pi \int_{0}^{\frac{\sqrt{2}}{2}} y^{2} d y$

This integrates to $\pi y\left[\begin{array}{l}\frac{\sqrt{2}}{2} \\ 0\end{array}\right.$, minus $2 \pi \cdot \frac{1}{3} \cdot y^{3}\left[\begin{array}{c}\frac{\sqrt{2}}{2} \\ 0\end{array}\right.$
$\pi\left(\frac{\sqrt{2}}{2}\right)-\pi(0)-\left[\frac{2}{3} \pi\left(\frac{\sqrt{2}}{2}\right)^{3}-\frac{2}{3} \pi(0)^{3}\right]$
$\frac{\sqrt{2}}{2} \pi-\frac{2}{3} \pi\left(\frac{\sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2}}{2 \cdot 2 \cdot 2}\right)$
$\frac{\sqrt{2}}{2} \pi-\frac{2}{3} \cdot \frac{2 \sqrt{2}}{2 \cdot 2 \cdot 2} \pi=\frac{\sqrt{2}}{2} \pi-\frac{\sqrt{2}}{6} \pi=\frac{3 \sqrt{2}}{6} \pi-\frac{\sqrt{2}}{6} \pi=\frac{2 \sqrt{2}}{6} \pi=\frac{\sqrt{2}}{3} \pi$.
The point of having you do exercises like this is to let you show off your new ability to calculate the volume of oddly shaped objects. This of course works only when those objects can be described by equations, which is the responsibility of the mathematical modeling people.

## 3. Cylindrical Shells

## Cut an infinitely thin cylindrical shell and lay it flat to see that its volume is $2 \pi r \mathrm{dr}$.

Using cylindrical shells, $V=\int 2 \pi r h d r$.
The integral starts at the center of the object, and goes out to the edge.
More than one function may determine the height of your shells.

Just like we can fill a shape with disks or washers, we can fill the entire volume with infinitely thin hollow cylinders. This method provides a definite advantage at times when it is difficult to calculate the maximum height of your object. If you don't know the maximum height of an object you couldn't use disks or washers because you wouldn't know how high to stack them. Using cylinders only requires you to know the width of the object.

If you have difficulty visualizing how to calculate the volume of a thin hollow cylinder, just make yourself one out of paper, and then flatten it out again. The volume is $2 \pi r$ (the circumference
of the cylinder) times the height times the very small thickness dr. The volume of each shell is an infinitely tiny part of the total volume $V$ :
$d V=2 \pi r h d r$
To find the volume $V$ we integrate:
$\int d V=\int 2 \pi r h d r$
Depending on how you stack your cylinders, dr may be dx or dy . Once you have decided that you will need say, dy, you must express both $r$ (the radius of each cylinder) and $h$ (the height of each cylinder) in terms of $y$ so that you have only a single variable in your integral.

## Example 1

Using slices, we found that the volume of a cylinder with a radius of 3 cm and a height of 10 cm is $90 \pi \mathrm{~cm}^{3}$. There is another way to find this volume using calculus. Recall that we can divide a circle into infinitely thin rings, and add those rings back up to get the area of a circle. The circular base of the cylinder can be divided up into rings with a thickness of dr, and a circumference of $2 \pi r$. The area of each infinitely thin ring is $2 \pi r d r$.

Just a word of caution here: the radius of the whole circle is 3 cm , but we are using $r$ as a variable to indicate the radius of each ring. Do not insert a value of 3 in place of $r$ in your integral.

Now, instead of just cutting up the base into rings, imagine cutting up the entire cylinder into infinitely thin hollow shells. To determine the volume of each hollow shell, I like to imagine cutting it open and laying it flat. The length of the resulting rectangle is the circumference: $2 \pi r$. The dimensions of this flattened shell are $2 \pi r \cdot d r \cdot 10$ for a volume of $20 \pi r d r$. Add up the volume of all the shells from $r=0$ to $r=3: \int_{0}^{3} 20 \pi r d r=10 \pi r^{2}\left[\begin{array}{l}3 \\ 0\end{array}=90 \pi \mathrm{~cm}^{3}\right.$.

## Example 2

Consider the area inside a circle with radius 1 centered at the origin, bounded by the line $y=x$ and the $x$-axis. Find the volume of the solid generated by rotating this area around the $y$-axis.


If this problem looks familiar, it is because we solved it before by using washers. However, the volume can also be found by filling it with cylindrical shells.

The shape in this problem can be filled with cylindrical shells that have their radius along the $x$ axis. The thickness of each shell is dx because we are setting our shells along the x -axis. Each shell has a volume dV, which equals $2 \pi r h$ dr

There are actually two parts to this. From $x=0$ to $x=\sqrt{2} / 2$, these shells have a radius of $x$ and height y , which is just x here. From $\mathrm{x}=\sqrt{2} / 2$ to $\mathrm{x}=1$ the radius is still x but the height y is now $\sqrt{1-\mathrm{x}^{2}}$.
$\int_{0}^{\frac{\sqrt{2}}{2}} 2 \pi x \cdot x d x+\int_{\frac{\sqrt{2}}{2}}^{1} 2 \pi x \sqrt{1-x^{2}} d x$

The second integral needs a u-substitution:
$u=1-x^{2} \quad$ If $x=\frac{\sqrt{2}}{2} u=\frac{1}{2}$, and if $x=1$ then $u=0$
$d u=-2 x d x$
$\int_{\frac{1}{2}}^{0}-\pi \sqrt{u} d u$
$\frac{2}{3} \pi u^{\frac{3}{2}}\left[\begin{array}{c}1 / 2 \\ 0\end{array}\right.$, which is $\frac{2}{3} \pi \sqrt{\left(\frac{1}{2}\right)^{3}}-\frac{2}{3} \pi(0)^{\frac{3}{2}}=\frac{2}{3} \pi \sqrt{\frac{1}{8}}=\frac{2}{3} \pi \frac{1}{\sqrt{8}}=\frac{2}{3} \pi \frac{1}{2 \sqrt{2}}=\frac{1}{3} \pi \frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{6} \pi$

Adding the results of both integrals gives us $\frac{\sqrt{2}}{6} \pi+\frac{\sqrt{2}}{6} \pi=\frac{\sqrt{2}}{3} \pi$.

## Example 3

Consider the area bounded by the function $y^{2}=-x+5$, the line $x=3$, and the $y$-axis. Use cylindrical shells to find the volume of the solid generated by rotating this area around the x -axis. Check your answer using slices.


When you use shells things can look more confusing. It helps to set your shells down on a level surface, which is this case is the $y$-axis. That means each shell will have a thickness of dy. From $y=0$ to the first intersect point, all of the shells will have a height of 3 . That intersect point occurs when $x=3$, so $y^{2}=-3+5$ or $y=\sqrt{2}$. After that the shells get shorter, with a height of $x$. We're done once the curve intersects the $y$-axis, which happens when $x=0: y^{2}=0+5$ or $y=\sqrt{5}$.

First integrate from $y=0$ to $y=\sqrt{2}$. The radius of each shell is $y$ and the height is 3 , so the volume of one shell is $2 \pi \sqrt{2} \cdot 3 \cdot d y .6 \pi \int_{0}^{\sqrt{2}} y d y=3 \pi y^{2}\left[\begin{array}{c}\sqrt{2} \\ 0\end{array}\right.$, which is $6 \pi$. Notice that this is just the volume of a cylinder with a radius of $\sqrt{2}$ and a height of 3 .

Next, we need to integrate the remaining shells. They will have a radius of $y$ and a height of $x$. The volume of one of those shells is $2 \pi y \mathrm{x} d \mathrm{y}$. Add the volumes from $\sqrt{2}$ to $\sqrt{5}$ :
$2 \pi \int_{\sqrt{2}}^{\sqrt{5}} y x d y$
$2 \pi \int_{\sqrt{2}}^{\sqrt{5}} y\left(5-y^{2}\right) d y$
$2 \pi\left(\int_{\sqrt{2}}^{\sqrt{5}} 5 y d y-\int_{\sqrt{2}}^{\sqrt{5}} y^{3} d y\right)$
$2 \pi\left(\frac{5}{2} y^{2}\left[\begin{array}{l}\sqrt{5} \\ \sqrt{2}\end{array}-\frac{1}{4} y^{4}\left[\begin{array}{l}\sqrt{5} \\ \sqrt{2}\end{array}\right)=2 \pi\left(\frac{5}{2} \cdot 5-\frac{5}{2} \cdot 2-\left(\frac{1}{4} \cdot 25-\frac{1}{3} \cdot 4\right)\right)=4.5 \pi\right.\right.$
So, we have a total volume of $6 \pi+4.5 \pi=10.5 \pi$.

Using slices, we can go along the $x$-axis. Each slice will have a radius $y$ and a thickness of $d x$, so the volume of one slice is $\pi y^{2} d x$. Integrate that from $x=0$ to $x=3: \pi \int_{0}^{3} y^{2} d x$. We can replace $y^{2}$ by $-x+5: \pi \int_{0}^{3}(5-x) d x$. That splits nicely: $\pi\left(\int_{0}^{3} 5 d x-\int_{0}^{3} x d x\right)$ :
$\pi\left(5 \times\left[\begin{array}{l}3 \\ 0\end{array}-0.5 x^{2}\left[\begin{array}{l}3 \\ 0\end{array}\right)\right.\right.$
$\pi(15-0.5 \cdot 9)=10.5 \pi$

## Practice

1. Use both slices and cylindrical shells to determine the volume of a cylinder with a height of 5 inches and radius of 2 inches.
2. Use calculus to find the volume of the solid generated by rotating the area between $f(x)=|x-5|$ and $y=3$ around the $x$-axis. Use geometry to check your answer.

## XI. Differential Equations

## Filling a Leaking Bucket?

Consider the net rate of change, and the initial condition.

There are many situations where something is entering a storage container, and simultaneously leaving it. This allows for infinite variations of the leaking bucket problem on exams.

## Example 1

Water is flowing into a 5 gallon bucket at a rate of 30 gallons per hour, and leaking out at a rate of 20 gallons per hour. If the bucket initially contained 3 gallons of water, how long will it take to fill?

This problem is easily solved by just thinking about it, but we will use calculus here to follow along with our common sense. The first thing to do is to create an expression that shows the net rate of change. If water is flowing in at a rate of 30 gallons per hour, and leaking out at a rate of 20 gallons per hour, there is a net inflow of $30-20=10$ gallons per hour. That looks too simple, so write it in calculus language as the change in volume: $\frac{\mathrm{dV}}{\mathrm{dt}}=30-20=10$.

Now, how long will it take to fill the bucket? Well, if $\frac{\mathrm{dV}}{\mathrm{dt}}=10$, and we know the initial volume of water, we can find the actual volume equation by taking an integral. You already know that if $\frac{\mathrm{dV}}{\mathrm{dt}}=10$, then V must be equal to $10 \mathrm{t}+\mathrm{C}$. You can write that nicely as shown below:
$\frac{d V}{d t}=10$
$d V=10 d t$
$\int d V=\int 10 d t$
$V=10 t+C$
Whenever we know the rate of change, the integral gives this kind of general expression for the actual value. At $t=0$, the volume is 3 . This is called an initial condition. Because you know this initial condition, you can determine the value of $C$, which is 3 in this case.

The final equation for the volume of water in the bucket is $\mathrm{V}=10 \mathrm{t}+3(0 \leq \mathrm{V} \leq 5)$, where t is measured in hours. It will take $\frac{1}{5}$ of an hour, or 12 minutes to fill the 5 gallon bucket with 2 more gallons, which makes perfect sense when the water is flowing in at a net rate of 10 gallons per hour. Actual exam problems will look more complicated, but they are based on this simple idea.

## Example 2

A recycling sorting facility that operates 6 days per week can process and ship out 19 tons of recyclables per day. During a typical week, material arrives at the plant at a rate modeled by the function $a(t)=-t(t-6)\left(0.2 t^{2}+1\right)$, where $t$ is in days. If there are 27 tons of unprocessed material at the plant when it opens Monday morning, how much will be left to process when the plant closes Saturday evening?

Here the net rate of change is inflow - outflow or $-\mathrm{t}(\mathrm{t}-6)\left(0.2 \mathrm{t}^{2}+1\right)-19$. The function that gives the actual amount of recyclables at the plant is the integral of that. Multiply out and take the integral: $\int\left(-0.2 \mathrm{t}^{4}+1.2 \mathrm{t}^{3}-\mathrm{t}^{2}+6 \mathrm{t}-19\right) \mathrm{dt}=-0.04 \mathrm{t}^{5}+0.3 \mathrm{t}^{4}-\frac{1}{3} \mathrm{t}^{3}+3 \mathrm{t}^{2}-19 \mathrm{t}+\mathrm{C}$.

When $t$ is zero, at the start of the week, the value of this function is 27 , so $C=27$. At $t=6$, the amount of material remaining will be 26.76 tons. Notice that we can get this same result by taking the definite integral from 0 to 6 , which is -0.24 , and adding the initial amount, 27. Essentially, this is taking the change and adding it up for 6 days to get the total change, and then adding what you started with. The definite integral is better here, because your graphing calculator will get it for you! [Note that although C cancels out when you take a definite integral, the initial condition does not actually disappear. You may need to add it depending on what it is you're trying to calculate.]

A graphing calculator will also tell you when the amount of recyclables at the plant is at a maximum or minimum, because the rate of change will be zero at these points.

## Separable Differential Equations

Separable differential equations can be solved by moving the $x$ and $y$ portions of the equation to opposite sides, and then integrating. If there are two terms on one side, combine them into a single term if possible.

Note that $\mathrm{e}^{(\mathrm{x}+\mathrm{c})}=\mathrm{e}^{\mathrm{x}} \cdot \mathrm{e}^{\mathrm{c}}=C \mathrm{e}^{\mathrm{x}}$

If you look back at the Implicit Differentiation section, you will see that we differentiated $y^{2}+x^{2}=100$ and got $\frac{d y}{d x}=-\frac{x}{y}$. The original function $y$ appears in this differential equation. $A$ differential equation is an equation that contains differentials, which are things like dy and dx . Mathematicians define a differential equation as an equation that shows a relationship between an unknown function and one or more of its derivatives. Implied in this definition is that the object of the game is to find the unknown function. So, if you start with $\frac{d y}{d x}=-\frac{x}{y}$, is it possible to work backwards and determine that $y= \pm \sqrt{100-x^{2}}$ ? Well, not quite, since some information disappeared when we took the derivative. $y^{2}+x^{2}=100$ has the same derivative as $y^{2}+x^{2}=0$, or $y^{2}+x^{2}=5$. We will have to add a constant back in, just as we do for any indefinite integral.

Before we try to solve $\frac{d y}{d x}=-\frac{x}{y}$, let's do some easier problems.

## Example 1

$\frac{d y}{d x}=2 x$
Hey, I know that! The solution is $y=x^{2}+C$. If only $x$ appears on the right side of the equation, we can solve it easily by taking the integral on both sides. To do that properly you should rewrite $\frac{d y}{d x}=2 x$ as $d y=2 x d x$. Then take the integral on both sides:
$\int d y=\int 2 x d x$
$y+C_{1}=x^{2}+C_{2}$

Combine the two constants into a single constant to get $y=x^{2}+C$.

## Example 2

$\frac{d y}{d x}=y$
Now all we have to do is find a function $y$ that has a derivative equal to itself. A good candidate for this is $\mathrm{y}=\mathrm{e}^{\mathrm{x}}$, for which the derivative is $\mathrm{e}^{\mathrm{x}}$. Notice that we cannot choose $\mathrm{y}=\mathrm{e}^{\mathrm{x}}+5$, because then the derivative, $\mathrm{e}^{\mathrm{x}}$, is no longer equal to y .

Let's do this nicely so we can see if there are any other possibilities. First we should rewrite the equation so we can take the integral on both sides:
$d y=y d x$
Oops, I can't find $\int y d x$ because there is more than one variable here. $y$ needs to move to the other side first:
$\frac{1}{y} d y=d x$
$\int \frac{1}{y} d y=\int d x$
$\ln |y|+C_{1}=x+C_{2}$
The In function accepts only positive values, so we need the absolute value sign for y in case it happens to be negative.

Just use a single constant that combines the constant from both integrals:
$\ln |y|=x+c$
$e^{\ln |y|}=e^{(x+c)}$
Be careful here, $\mathrm{e}^{(\mathrm{x}+\mathrm{c})}$ has exponents that are added, which is the result of the multiplication $\mathrm{e}^{\mathrm{x}} \cdot \mathrm{e}^{\mathrm{c}}$.
$|y|=e^{x} \cdot e^{c} \quad y$ could be negative, but $e^{x} \cdot e^{c}$ is always positive
$y= \pm e^{x} \cdot e^{c}$

Notice that $e^{c}$ is just some constant, so we can call that $C$. Once we replace $e^{c}$ by a regular constant we no longer need the $\pm$ sign, since $C$ can be negative.
$y=C e^{x}$ is the solution to the differential equation $\frac{d y}{d t}=y$.
This says that the derivative of $\mathrm{y}=\mathrm{Ce}^{\mathrm{x}}$ is $\mathrm{Ce}^{\mathrm{x}}$, which we really already knew.

## Example 3

Now let's try $\frac{d y}{d x}=-\frac{x}{y}$
Rearrange the equation to $y d y=-x d x$
$\int y d y=\int-x d x$
$\frac{1}{2} y^{2}=-\frac{1}{2} x^{2}+C_{1}$
$y^{2}=-x^{2}+C$
$y^{2}=C-x^{2}$
$y= \pm \sqrt{C-x^{2}}$. This is the general form of our original equation $y= \pm \sqrt{100-x^{2}}$.
As you solve various problems, you will use a given initial condition to turn your general equation into a specific solution. For example, if your solution is $y= \pm \sqrt{C-x^{2}}$, a given initial condition of $y=10$ when $x=0$ will allow you to find that $C=100$, and the $\pm$ sign can be removed:
$y=\sqrt{100-x^{2}}$

## Example 4

Find an equation for the volume $V$, given that $\frac{d V}{d t}=75+\frac{V}{10}$ and the volume is 50 at $t=0$.

Well, I can separate $d V$ and $d t$ like this: $d V=75 d t+\frac{V}{10} d t$. Then I move $V$ to the other side .... Hmm , that won't work out so well. When I try to divide both sides by V , I get $\frac{75 \mathrm{dt}}{\mathrm{V}}+\frac{1}{10} \mathrm{dt}$ on the right. I could move 75 dt to the left, but that leaves dt on the wrong side

The best thing to do in a situation like this is to create a single fraction on the right:
$d V=\frac{750}{10} d t+\frac{V}{10} d t$
$d V=\frac{750+V}{10} d t$
Now I can divide both sides by $750+\mathrm{V}$ :
$\frac{1}{750+V} d V=\frac{1}{10} d t$
$\int \frac{1}{750+V} d V=\int \frac{1}{10} d t$
$\ln |750+\mathrm{V}|=\frac{1}{10} \mathrm{t}+\mathrm{c}$
$750+V= \pm \mathrm{e}^{\mathrm{t} / 10+\mathrm{c}}$
$V= \pm e^{c} e^{t / 10}-750$
$V=\mathrm{Ce}^{\mathrm{t} / 10}-750$
If $\mathrm{V}=50$ when $\mathrm{t}=0$, then $\mathrm{Ce} \mathrm{e}^{0 / 10}$ must be equal to 800 . Because $\mathrm{e}^{0}$ is just 1 , we can say that C = 800:
$V(t)=800 e^{t / 10}-750$

Make sure to check your work:
$\frac{\mathrm{dV}}{\mathrm{dt}}=75+\frac{\mathrm{V}}{10}=75+\frac{800 \mathrm{e}^{\frac{t}{10}}-750}{10}=80 \mathrm{e}^{\mathrm{t} / 10}$

## Slope Fields

A slope field is a visual representation of a differential equation $f(x, y)$. The slopes of the line segments show the value of the derivative of the original function $F$ for selected values of $x$ and $y$.

Sketch the graph of F by starting at the point given by the initial condition, and follow the rate of increase or decrease shown by the slope of the lines.

Even if you can't solve a differential equation, you can create a slope field. This is a confusinglooking drawing with a whole bunch of little lines on it. It is created by solving the differential equation for random values of $x$ and $y$, and then indicating the solution by drawing a very short line segment with that particular slope at the point ( $\mathrm{x}, \mathrm{y}$ ). If y does not appear in the differential equation, you just calculate the slope for a random value of $x$, say $x=3$. Then you place little line segments with the calculated slope at all points $(3, y)$ on your drawing.

Let's consider a very simple differential equation: $\frac{d y}{d x}=2 x$. For all integer values of $x$ and $y$ between -2 and 2 , find the value of $d y / d x$ at ( $x, y$ ). Now create a slope field by drawing small lines with the correct slope. As you are working, you will notice that the slopes are the same at many of the points. Next, draw the slope field for $\frac{d y}{d x}=x y$. By actually drawing these slope fields yourself, you should get a feel for the difference between fields that depend only on $x$, and those that depend on both $x$ and $y$. For exams, it is important to be able to match slope fields with the correct differential equation.

Notice that drawing a slope field is not particularly difficult, but it is tedious work. For this reason, slope fields are best made by computers, not by people. There are many online slope field generators available. The picture below shows a slope field for $\frac{d y}{d x}=2 x$ :

## 

## Slope Field Calculator



Notice that the little lines appear to form a parabola-like pattern. The x and y coordinates were selected to range from -1 to 1 . If you set the initial conditions to $x=0, y=0$, the computer will draw the simplest original function, the parabola $y=x^{2}$.

It is also possible for a human being to use a slope field to make a sketch of one of the functions that is the solution to a differential equation. You need a starting point, also called an initial condition, so you know which solution you are drawing. From your starting point, you follow the directions indicated by the little lines you encounter. At first it may seem like that could lead you anywhere, but with some practice you will see that there is really only one way to go from any particular point. Again, a computer can perform this task much more accurately and efficiently.

## Taking the Derivative of a Differential Equation

If a differential equation contains $y$ or another function, use implicit differentiation!

Because this could get complicated, we'll create our own simple example. Let's pick a function with a simple derivative, yet have it contain the original function within that derivative so the differential equation has y in it . I'll use the simple function $\mathrm{y}=\mathrm{e}^{2 \mathrm{x}}-5$. The derivative of this is just $2 e^{2 x}$, so I can say that $\frac{d y}{d x}=2 y+10$. Now, let's take the derivative of that, which will be the second derivative of $y$.

I know it is tempting to just put 2 on the right side, but be careful, y is a function rather than just a variable! Use implicit differentiation: $\frac{d^{2} y}{d x^{2}}=2 \frac{d y}{d x}$

Because we already have an expression for $\frac{d y}{d x}$, we can just substitute that into the equation. That means that the second derivative $\frac{d^{2} y}{d x^{2}}$ is $2(2 y+10)$, which is $4 y+20$. We also have an expression for $y$ in this case, so we can say that $4 y+20=4\left(e^{2 x}-5\right)+20=4 e^{2 x}$.

Check that this answer is correct by differentiating $y=e^{2 x}-5$ twice.

## Example

Find the derivative of $\frac{d P}{d t}=\frac{1}{10} \mathrm{P}^{2}-100$, with respect to t .
Notice that P is a function of t , so use implicit differentiation:
$\frac{\mathrm{d}^{2} \mathrm{P}}{\mathrm{dt}^{2}}=\frac{1}{10} 2 \mathrm{P} \frac{\mathrm{dP}}{\mathrm{dt}}$
$\frac{\mathrm{d}^{2} \mathrm{P}}{\mathrm{dt}^{2}}=\frac{1}{5} \mathrm{P}\left(\frac{1}{10} \mathrm{P}^{2}-100\right)$
$\frac{\mathrm{d}^{2} \mathrm{P}}{\mathrm{dt}^{2}}=\frac{1}{50} \mathrm{P}^{3}-20 \mathrm{P}$

## Solving Differential Equation Problems

If you have a differential equation describing $f^{\prime}(x)$, and one point on the graph of the initial function $f(x)$, you may need to find a second point on the graph of $f(x)$. Here you only have a few options available to you based on what you have learned in your course.

1. It is possible to solve the differential equation using the skills you have acquired in your course. You can then use the given point ( $\mathrm{x}, \mathrm{f}(\mathrm{x})$ ) to eliminate the constant and find the specific equation for $f(x)$.
2. The differential equation is too hard for you to solve, but your calculator is able to find the corresponding definite integral, $\int_{a}^{b} f^{\prime}(x) d x$. Use the Fundamental Theorem of Calculus to see that $\int_{a}^{b} f^{\prime}(x) d x=f(b)-f(a)$. If you know $f(a)$ you can find $f(b)$ because the calculator will give you the numerical value of the integral.
3. Neither you nor your calculator can do anything to solve the differential equation. You can still use linear approximation to estimate the value of the function $f(x)$ at a point that is reasonably close to the given point. This is usually fairly easy because the differential equation provides the slope of the tangent line at the given point. To make things a little harder you may be asked to figure out if your estimate is above or below the actual value. This means that you must know if $f(x)$ is concave up or concave down over the entire interval between the two points. Check the second derivative, $\frac{d^{2} y}{{d x^{2}}^{2}}$. If both $x$ and $y$ appear in the second derivative, you will need to know if they are positive or negative over the interval so you can determine the sign of the second derivative. Recall that if $\frac{d^{2} y}{d x^{2}}$ is positive the function is concave up, and your estimate will be lower than the actual value.
4. You can't solve the differential equation and the given point is not very close to the point where you need the value. You can draw a slope field to estimate the value of the function at the required point. This is slow and tedious and very unlikely to appear on your exam.
